COEFFICIENTS AND NON-TRIVIALITY OF THE JONES POLYNOMIAL

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Abstract. This is a research-expository style transcript of my talk at KIAS, Seoul, Korea, September 25, 2016, for the audience of the seminar.

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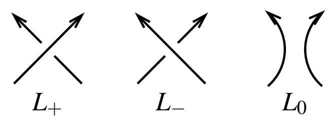
1. Introduction

The *Jones polynomial* V (now commonly used with the convention of [J]) is a Laurent polynomial in one variable t of oriented knots and links, and can be defined by being 1 on the unknot and the *(skein) relation*

$$t^{-1}V(L_+) - tV(L_-) = -(t^{-1/2} - t^{1/2})V(L_0)\,. \tag{1}$$

Herein $L_{\pm,0}$ are three links with diagrams differing only near a crossing.



(2)

When

$$V_K = V_0 t^k + V_1 t^{k+1} + \ldots + V_d t^{k+d} \tag{3}$$

with $V_0 \neq 0 \neq V_d$ is the Jones polynomial of a knot or link K, we will use throughout the paper the notation $V_i = V_i(K) = V_i$ and $\bar{V}_i = \bar{V}_i(K) = V_{k-i}$ for the the i-th or i-th last coefficient of V, and will write for d the *span* $\operatorname{span} V_K$ of V, for k the *minimal degree* $\min\deg V_K$ and for $k+d$ the *maximal degree* $\max\deg V_K$.

For quite a while one is wondering what topological information the Jones polynomial contains, and in connection with this, one posed the

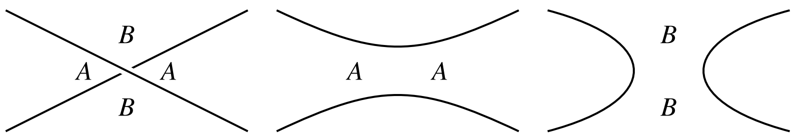
Question 1 Does there exist a non-trivial knot with trivial Jones polynomial?

While the existence of non-trivial links with trivial polynomial is now settled for links of two or more components by Eliahou-Kauffman-Thistlethwaite [EKT], the (most interesting) knot case remains open. The question remains unanswered, though some classes of knots have been excluded from having trivial Jones polynomial. These results are obtained in [Ka, Mu, Th2] for alternating knots, [LT] for adequate knots, [St2] for positive knots, and also in [Th2] for the Kauffman polynomial of semiadequate knots. Except for these (meanwhile classical) results, and despite considerable (including electronic) efforts [Bi, Ro, DH, St5], even nicely defined general classes of knots on which one can exclude trivial polynomial are scarce. (I came across some work of Yamada who stated that he verified all knots up to 21 or 22 crossings, but I have no reference to it.)

1.1. Semiadequacy and Kauffman bracket

It is useful to define here the Jones polynomial via Kauffman's state model. We thus consider the bracket [Ka] (rather than Tutte, as Dasbach-Lin) polynomial.

Below are depicted the A- and B-corners of a crossing, and its both splittings. The corner A (resp. B) is the one passed by the overcrossing strand when rotated counterclockwise (resp. clockwise) towards the undercrossing strand. A type A (resp. B) splitting is obtained by connecting the A (resp. B) corners of the crossing.



(4)

Recall, that the Kauffman bracket $\langle D\rangle$ of a link diagram D is a Laurent polynomial in a variable A, obtained by summing over all states S the terms

$$A^{\#A(S)-\#B(S)}\left(-A^2-A^{-2}\right)^{|S|-1}, \tag{5}$$

where a *state* is a choice of *splicings* (or *splittings*) of type A or B for any single crossing (see (4)), $\#A(S)$ and $\#B(S)$ denote the number of type A (respectively, type B) splittings and $|S|$ the number of (disjoint) circles obtained after all splittings in S. We call the *A-state* the state in which all crossings are A-spliced, and the *B-state* is defined analogously.

The Jones polynomial of a link L can be specified from the Kauffman bracket of some diagram D of L by

$$V_L(t) = \left(-t^{-3/4}\right)^{-w(D)} \langle D\rangle\Big|_{A=t^{-1/4}}, \tag{6}$$

with $w(D)$ being the writhe of D.

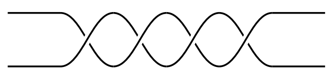
Let $X \in \mathbb{Z}[t,t^{-1}]$. The *minimal* or *maximal degree* $\min\deg V$ or $\max\deg V$ is the minimal resp. maximal exponent of t with non-zero coefficient in V. Let $\operatorname{span}_t V = \max\deg_t V - \min\deg_t V$. The coefficient in degree d of t in V is denoted $[V]_{t^d}$ or $[V]_d$. We will use more commonly another notation for coefficients.

Definition 1 Let $V \in \mathbb{Z}[t^{\pm 1}]$ or $V \in t^{\pm 1/2} \cdot \mathbb{Z}[t^{\pm 1}]$, and $n \geq 0$ an integer. Let $m = \min\deg V$ and $M = \max\deg V$ (then $2m \in \mathbb{Z}$). We write $V_n(L) := [V]_{m+n}$ and $\bar{V}_n(L) := [V]_{M-n}$ for the $n+1$-st or $n+1$-last coefficient of V.

2. The second coefficient

For the Jones polynomial of special types of knots, more is known.

The *twist number* $t(L)$ of a link is the minimal twist number $t(D)$ of all diagrams D, where $t(D)$ is the number of pieces in D like



It occurred in recent work of Lackenby-Agol-Thurston [La].

In [DL] Dasbach-Lin gave a description of the twist numbers of alternating diagrams by means of the second coefficient of their Jones polynomial. They considered $T_i(K) := |V_i| + |\bar{V}_i|$ and proved

Lemma 1 ([DL]) For an alternating knot diagram D, we have $t(D) = T_1(D)$.

They were motivated by

Question 2 What are the relations between volume and V?

Some recent excitement was caused by the Volume conjecture [MM]), which states that some complicated colored Jones polynomial values converge to the Gromov norm of the knot complement (= hyperbolic volume of all hyperbolic parts in the JSJ decomposition = hyperbolic volume for hyperbolic knots).

This conjecture seems, unfortunately, little helpful to determine the volume in practice. So we may ask: if we sacrifice '=' for a '$\leq$', are there more tangible and practical ways to relate volume to V?

Using Lemma 1 and the recent work of Lackenby-Agol-Thurston [La], Dasbach-Lin obtained certain relations between coefficients of the Jones polynomial and hyperbolic volume.

Corollary 1 ([DL]) For any alternating knot K, we have

$$C(T_1(K) - 1) \leq \operatorname{vol}(K) \leq C' T_1(K) \tag{7}$$

for some positive constants C, C'.

In fact, we have a qualitative improvement of the Dasbach-Lin result, stating that

Theorem 1 Every coefficient V_i of the Jones polynomial gives rise to a(n increasing) lower bound for the volume of alternating knots.

The previous occurrence of the second coefficient of the Jones polynomial in a different situation in [St] motivated the quest for understanding $V_1, \bar{V}_1$ in a broader context.

We return to the Kauffman bracket polynomial.

The concept of an adequate link was introduced by Lickorish and Thistlethwaite in [LT] to help determining the crossing number of certain links. Adequacy consists of the combination of two weaker properties called jointly semiadequacy. They are defined as follows.

We use the splicings from (4). One says a diagram D is *A-adequate* if the number of loops obtained after A-splicing all crossings of D is more than the number of loops obtained after A-splicing all crossings except one. Similarly one defines the property B-adequate. Then we set

$$\begin{aligned} \text{adequate} &= A\text{-adequate } \textit{and } B\text{-adequate}, \\ \text{semiadequate} &= A\text{-adequate } \textit{or } B\text{-adequate}, \end{aligned}$$

We call a link adequate resp. (A/B/semi)-adequate if it has an adequate resp. (A/B/semi)-adequate diagram.

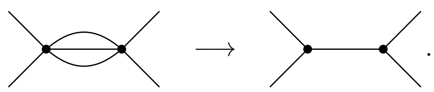
Note that semiadequate links are a much wider extension of the class of alternating links than adequate links. For example, only 3 non-alternating knots in Rolfsen's tables [Ro2, appendix] are adequate, while all 55 are semiadequate.

An alternative way to understand A-adequacy is to keep the trace of the crossings after each splitting. Then we have each of the traces of the crossings joining two loops, obtained after the splittings. The property A-adequate means that, in the set of loops obtained by A-splitting all crossings, each crossing connects two different loops. We call this set of loops the *A-state* of the diagram.

A basic observation in [LT] is that when L is A- resp. B-adequate then $|V_0(L)| = 1$ resp. $|\bar{V}_0(L)| = 1$. Thus if L is adequate, and in particular alternating, both properties hold.

In the following, we shall explain the second coefficient of the Jones polynomial in semiadequate diagrams. Bae and Morton [BMo] and Manchon [Mn] have done work in a different direction, and studied the extreme coefficients of the bracket (which are ± 1 in semiadequate diagrams) in more general situations.

Let $v(G)$ and $e(G)$ be the number of vertices and edges of a graph G. Let $\overline{G}$ be G with multiple edges removed (so that a simple edge remains).



We call $\overline{G}$ the *reduction* of G. Let $A(D)$ be the *A-graph* of D, a graph with vertices given by loops in the A-state of D, and edges given by crossings of D. (The trace of each crossing connects two loops.)

So a link diagram D is A-adequate, if $A(D)$ has no edges connecting the same vertex. (Anything with B is analogous.)

Theorem 2 ([LT]) If D is A-adequate then $V_0 = \pm 1$. If D is B-adequate then $\bar{V}_0 = \pm 1$. If D is adequate then $V(D) \neq 1$.

Now we have

Theorem 3 If D is A-adequate then $|V_1| = b_1(\overline{A(D)})$ is the first Betti number (number of cells) of the reduced A-graph. Similarly if D is B-adequate then $|\bar{V}_1| = b_1(\overline{B(D)})$.

<u>Key observation:</u> If $b_1(\overline{A(D)}) = 0$, then D admits a positive orientation, i.e., can be oriented so that all crossings become as L_+ in (2).

Corollary 2 No (non-trivial) semiadequate knot has $V = 1$.

Proof. If $V = 1$ then $V_1 = 0$, so the knot must be positive, but no non-trivial positive knot has $V = 1$. □

Actually: There is no non-trivial semiadequate link with trivial Jones polynomial (i.e., polynomial of the same component number unlink), even up to units $\pm t^k$.

3. Some (more) applications

3.1. Whitehead doubles

Untwisted Whitehead doubles have trivial Alexander polynomial, and are one suggestive class of knots to look for trivial Jones polynomial. (Practical calculations have shown that the coefficients of the Jones polynomial of Whitehead doubles are absolutely very small compared to their crossing number.)

Proposition 1 Let K be a semiadequate non-trivial knot. Then the untwisted Whitehead doubles $Wh_\pm(K)$ of K (with either clasp) have non-trivial Jones polynomial.

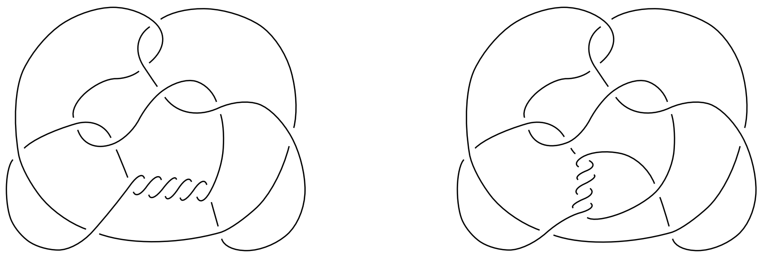
(Because V determines the degree-2-Vassiliev invariant v_2, among Whitehead doubles only untwisted ones may have trivial Jones polynomial.)

This generalizes a result for adequate knots in [LT] and positive knots in [St2] and considerably simplifies the quest for trivial polynomial knots among Whitehead doubles. One can combine this condition with the previous ones, the vanishing of the Vassiliev invariants of degree 2 and 3 on K (see [St2, St5]), to extend the verification of [St5] and establish that no non-trivial knot of ≤ 16 crossings has untwisted Whitehead doubles with trivial Jones polynomial.

3.2. Minimal positive diagrams

We exhibited two types of infinite families of positive knots with no minimal (crossing number) positive diagram. The first example of such a knot was given in [St6]. We obtained two type of generalizations of this example.

Proposition 2 ([St4]) There are infinitely many positive knots that have no minimal positive diagram, which are (a) of genus 3 and (b) fibered.



(*a*) (*b*)

3.3. *k*-almost positive knots

Using help from Bae and Morton [BMo] (and more techniques, too long to discuss here), we got this. The proof is long and given in my book [St7].

Proposition 3 A k-almost positive knot has non-trivial Jones polynomial for $k \leq 3$ and non-trivial skein polynomial for $k \leq 4$.

3.4. 3-braids

We call a braid word semiadequate (A-, B-adequate, adequate = etc.) if the closure diagram is semiadequate (etc.). A braid is semiadequate (etc.) if it has a semiadequate (etc.) word.

Thistlethwaite's work [Th] implies that if β is a semiadequate (etc.) braid then its semiadequate (etc.) words are of minimal (Artin generator) length in the conjugacy class of β (i.e. also for all braids conjugate to β). The interesting feature of 3-braids is that the converse holds for semiadequacy:

Theorem 4 A minimal length word in any 3-braid conjugacy class is semiadequate.

(One can also explicitly describe such words algebraicly.)

Corollary 3 3-braid links are semiadequate, and so have non-trivial Jones polynomial up to units.

This result is, for knots, originally due to Takahashi [Ta] (little known, and often imprecisely mentioned, e.g., in [B]). It was known that the Burau representation determines the Jones polynomial for 3 and 4-braids [J]. There is a direct connection between the faithfulness of the Burau representation and the lack of knots with trivial polynomial. In fact, one can use the above corollary to give another proof of the faithfulness for $n = 3$. Bigelow [Bi] is hoping(?) that 4-strand Burau many not be faithful, and is challenging the computers with this idea to find a $V = 1$ knot among closed 4-braids. The closed 4-braid $(\sigma_2\sigma_1\sigma_3\sigma_2)^2\sigma_1^3\sigma_3^{-3}$ (which is among the links given in [EKT]), however, has trivial polynomial up to units. So it cautions about attempts to understand the (possible) non-existence of trivial polynomial knots among 3- or 4-braids in terms of the (possible) faithfulness of the Burau representation. Our proof for 3-braids has indeed little to do with Burau. By the above example, our result also fails for 4-braids.

Combining braid semiadequacy with work in [St3, BM, Xu], we can actually classify all 3-braid links with given Jones polynomial. In particular, we know that

Corollary 4 There are only finitely many closed 3-braids with the same Jones polynomial.

This was known to be true for the skein polynomial [St3]. The links of Traczyk [Tr] show that this is not true for Jones polynomials up to units, and by connected sum for fixed polynomials on 5-braids. (The status of 4-braids here remains unclear.) Also Kanenobu [K2] constructed finite families of 3-braids of any arbitrary size, so that our result is the maximal possible.

The corollary implies the existence of some upper bound on the volume in terms of the Jones polynomial. We can make an estimate more concrete:

Corollary 5 If K is a 3-braid link, which is not a closed positive or negative 3-braid, then $\mathrm{vol}(K) \le C' \cdot T_1$ (similar to Dasbach-Lin's (7)).

Futer-Kalfagianni-Purcell [FKP] have more recently proved a lower bound (much harder!).

3.5. Montesinos links

Corollary 6 Montesinos links are semiadequate. So no Montesinos link has trivial Jones polynomial up to units.

Also the following is true:

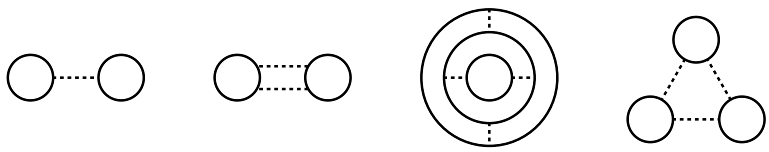
Proposition 4 There exists an upper bound on the volume of a Montesinos link in terms of the Jones polynomial.

For an explicit bound, however, I must involve T_2, and prove a formula similar to theorem 3 for V_2. It depends on more than $A(D)$.

4. The third coefficient

When L is an A-adequate link, then $V_n(L)$ for $n \le 2$ were studied in [DL, DL2, St4]. We recall the formulas briefly. Let D be an A-adequate diagram of L. (We will assume that D is connected.)

Let $|A(D)|$ be the number of loops in the A-state $A(D)$, and the quantities e, e_{++}, δ, and $\triangle$ be the number of pairs or triples of loops in $A(D)$ for which there exist crossing traces (obtained as in figure 4) making them look (up to moves in S^2) like in:



e e_{++} δ $\triangle$

(We do not require that these be the only traces connecting the loops, only that such traces should exist.) Note that when D is alternating, then $\delta = 0$.

Then, theorem 3 states that

$$|V_1(L)| = e - |A(D)| + 1\,. \tag{8}$$

Define the *intertwining graph* $IA(D)$ of the A-state of D as follows. Vertices of $IA(D)$ are multiple edges in the A-state of D (counted by e_{++}) and edges in $IA(D)$ go between (vertices for) intertwined multiple edges in the A-state of D (counted by δ). Thus

$$\chi(IA(D)) = e_{++} - \delta.$$

Note that $IA(D)$ is not determined by $A(D)$, but that $\triangle = \triangle(A(D))$ is, for it counts the cycles of length 3.

We can state now the formula for $V_2(L)$.

Theorem 5 ([DL2, St4])

$$|V_2(L)| = \binom{|V_1(L)|+1}{2} + e_{++} - \delta - \triangle = \binom{|V_1(L)|+1}{2} + \chi(IA(D)) - \triangle. \qquad (9)$$

It is immediately clear, that $\chi(IA(D)) - \triangle$ must be an invariant of the A-adequate diagram D, but using a cabling argument, we can even separate them.

Corollary 7 The numbers $\chi(IA(D))$ and $\triangle(A(D))$ are invariants of the A-adequate diagram D of the same link L.

We have thus now extracted three semiadequacy invariants: $\chi(\overline{A(D)})$ (or, equivalently, $b_1(\overline{A(D)})$), $\chi(IA(D))$ and $\triangle(A(D))$. We know that for all A-adequate diagrams D of the same link L, each invariant is the same.

5. Further applications

We talked about Proposition 4 as one application.

Another one is the extension of the oriented amphicheiral 3-braid link classification of Birman-Menasco [BM].

Theorem 6 Let L be a 3-braid link which is unorientedly amphicheiral. Then either

(a) it is orientedly amphicheiral, in which case it is the closure of an alternating 3-braid with Schreier vector admitting a dihedral (anti)symmetry, or

(b) it is one of the following links:

- a Hopf link,
- a Hopf link with a split trivial component,
- the connected sum of two Hopf links of the same sign,
- the (3,3)-rational link (or (1,1,1,3)-pretzel link), or
- the link 9^2_{61} (the closure of $(\sigma_1\sigma_2^2\sigma_1)^2\sigma_2^{-1}$; see Figure 1.66.1 in [Ki, p. 46]).

The main highlight of this whole story, though, is the construction of odd crossing number amphicheiral knots. This is a problem which goes back to Tait's tabulating work in the 1880's, and is essentially as old as knot theory itself!

Theorem 7 For each odd natural number $n \geq 15$, there exists an amphicheiral knot of crossing number n.

Similarly to Perko's knot, the first odd crossing number amphicheiral knot was found accidentally: Hoste and Thistlethwaite, in the course of routine knot tabulation, discovered an amphicheiral 15 crossing knot. (Their compilational work had previously shown that there are no amphicheiral knots of odd crossing numbers up to 13.) Settling the other crossing numbers is a major problem, though, since exhaustive enumeration is no longer a feasible approach – we face the well-known difficulty that we do not know (generally) how to determine the crossing number. A few other methods are known, but all they fail on such examples. Thus the way to our result is rather far, and below we will conclude by giving a brief outline of the proof.

Our three semiadequacy invariants become, also joined by a relative obtained from the Kauffman polynomial and Thistlethwaite's results [Th], the main tool for the proof of theorem 7. For given odd $n \geq 15$, we start with an amphicheiral knot K that has an n crossing diagram, which is semiadequate. Luckily, such examples can be obtained by leaning on Hoste-Thistlethwaite's knot. The work in [Th] shows then that the crossing number of K is at least $n-1$, and were it $n-1$, a minimal crossing diagram D would be adequate. Then we have 4 invariants for both A-adequacy and B-adequacy each available. A detailed study of how an $n-1$ crossing diagram with such invariants must look like is necessary to exclude most cases for D. Hereby, among the various generalizations of Thistlethwaite's knot, one must choose carefully the one whose invariants make the exclusion argument most convenient (or better to say, feasible at all). Only a small fraction of possibilities for D remain, which are easy to check, and rule out, by computer. This allows us to conclude that in fact D cannot exist.

6. Open problems

Apart from the fundamental problem in Question 1, one can ask few other things.

Problem 1 Are there only finitely many semiadequate knots with the same Jones polynomial?

We saw this resolved for 3-braid knots (Corollary 4). But it remains unsettled more generally, even just asking about Montesinos knots. Kanenobu [K] constructed infinite families of knots with the same Jones polynomial, but they are not semiadequate. He also asked the following question, which seems useful to reiterate here.

Problem 2 (Kanenobu) Are there only finitely many knots with the same Kauffman polynomial?

One more topic to think about may naturally be this.

Problem 3 Understand coefficient 4, i.e., $V_3(D)$ for an A-adequate diagram D.

This looks hard, but perhaps not hopeless. At least in the case of positive braids, it was possible to see some information in V_3 in [St].

감사합니다!

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