On dual triangulations of surfaces

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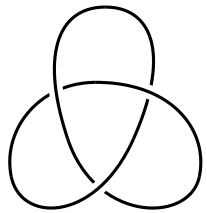
Contents

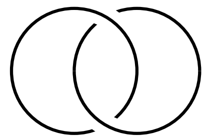
1. Canonical genus bounds hyperbolic volume

knot K $\quad S^1 \hookrightarrow S^3$

link L $\quad \underbrace{S^1 \cup \ldots \cup S^1}_{n \text{ components}} \hookrightarrow S^3$



$K =$ trefoil

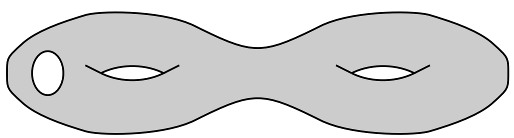


$L =$ Hopf link

Let $g(K)$ be the *genus* of K, given by

$$g(K) \;=\; \min\{\, g(S) \,:\, S \text{ is a Seifert surface of } K \,\},$$

where a Seifert surface S of K is as

 $= S \hookrightarrow S^3, \qquad \partial S = K\,,$

and its genus is $g(S) \;=\; \#\,\text{holes of } S$.

$g_f(K)$, the *free genus*, minimal genus of free surfaces S (i.e. $S^3 \setminus S$ a handlebody).

$g_c(K)$, the *canonical genus*, is the minimal genus of canonical surfaces S (obtained by Seifert's method).

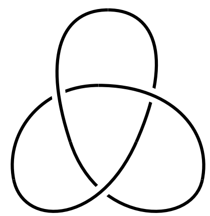
Canonical surfaces are free $\Rightarrow$

$$g(K) \leq g_f(K) \leq g_c(K)\,.$$

These are often (though not always) equalities. In particular:

Definition 1. alternating knots and links

A knot (or link) is *alternating* if it has a diagram where (along each component) one passes strands under-over.



Theorem 2 (Crowell-Murasugi '59-'61). *L alternating knot or link $\Rightarrow$ canonical surface of alternating diagram is of minimal genus ($\Rightarrow g(L) = g_c(L)$)*

(W.) Thurston: most knots (and links) L are *hyperbolic*:

$$S^3 \setminus L = \underset{\substack{\text{hyperbolic} \\ \text{3-space}}}{\underset{\uparrow}{H^3}} \Big/ \underset{\substack{\text{group of} \\ \text{isometries}}}{\underset{\uparrow}{\Gamma}},$$

and volume is finite: *volume of* L, $\mathrm{vol}(L)$.

(Convention: $\mathrm{vol}(L) := 0$ when L not hyperbolic.)

Theorem 3 (Brittenham).

$$\sup\{\,\mathrm{vol}(K)\,:\,g_c(K) = g\,\} < \infty\,. \tag{1}$$

Remark 4. But

$$\sup\{\,\mathrm{vol}(K)\,:\,g_f(K) = g\,\} = \infty$$

(B. $g = 1$, S. $g \geq 2$), thus maximal volume makes little sense for (free) genus. But it does for alternating knots, and sup is the same as (1).

For <u>links</u> use (canonical) *Euler characteristic* χ, χ_c. (knots: $\chi_{(c)} = 1 - 2g_{(c)}$)

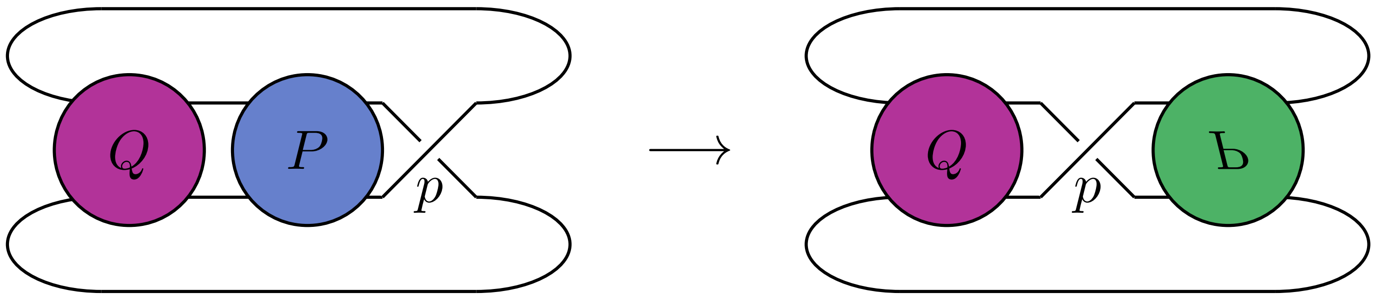
Let

$$v_\chi \;=\; \sup\{\,\mathrm{vol}(L)\;:\;\chi_c(L) = \chi\,\} \tag{2}$$

Computation? Estimates? Will return to this after a (long) detour.

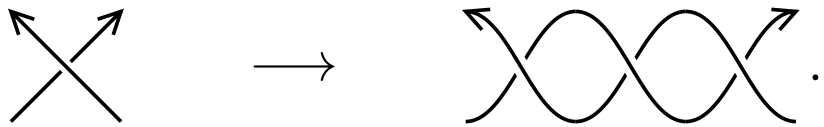
2. Crossing equivalence, generators

A *flype* is the move



Definition 5. A $\bar{t}_2'$ *move* is a move creating a pair of crossings reverse twist

($\sim$-)equivalent to a given one:



Alternating diagram *generating*: $\iff$ irreducible under flypes and reverse of $\bar{t}'_2$ moves. For such diagram D,

$$\textit{(generating) series of } D := \left\{ \begin{array}{c} \text{diagrams obtained by flypes} \\ \text{and } \bar{t}'_2 \text{ moves on } D \end{array} \right\}.$$

generator:= alternating knot whose alternating diagrams are generating.

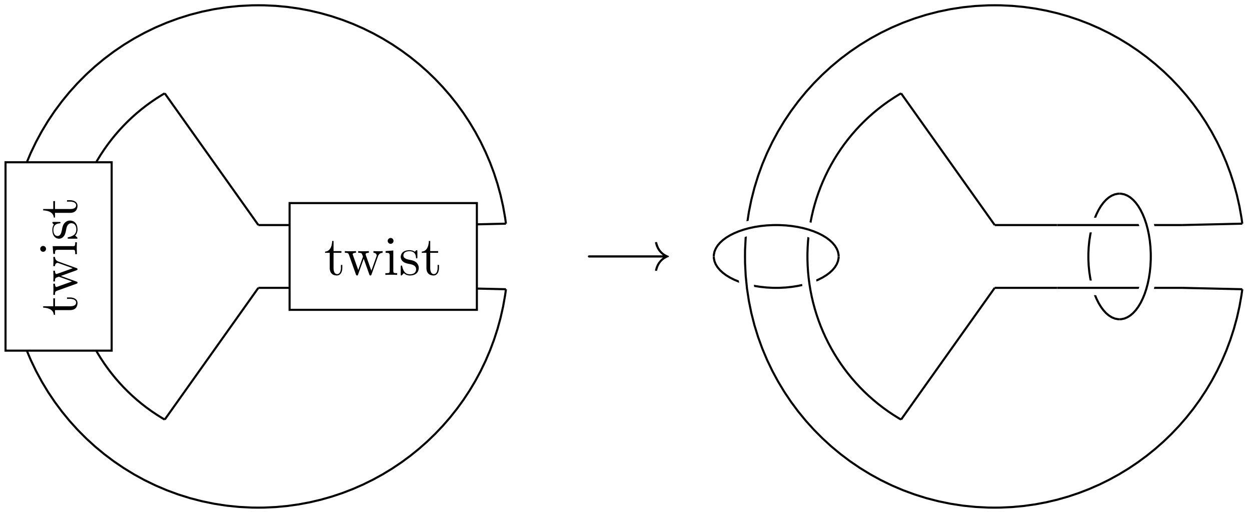
Theorem 6 (S; Brittenham). *The number of generators of given genus is finite.*

More precisely (S.): they have $\leq 6g-3$ $\sim$-equivalence classes. (For links -3χ, except $\chi = 0$, where the Hopf link is the only generator.)

Definition 7. Call a generator *even/odd* if it has even/odd crossings. Call it *maximal* if it has $6g-3$ $\sim$-equivalence classes.

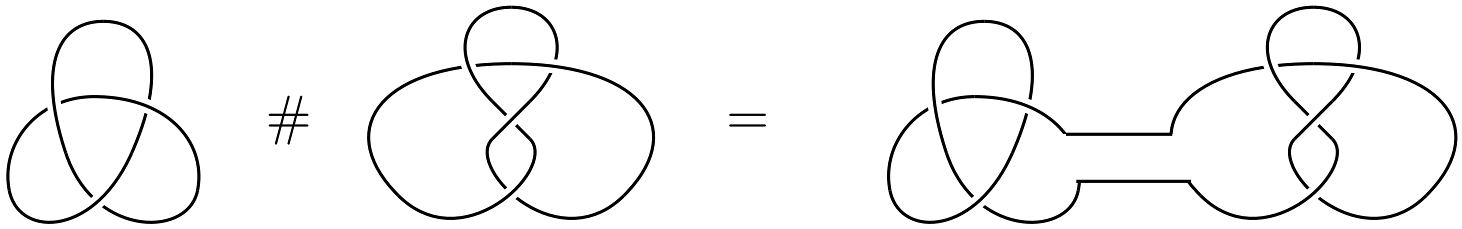
Proof of theorem 3. By Thurston's hyperbolic surgery theorem $\Rightarrow$

$$\sup(\text{series}) = \mathrm{vol}(\underline{\text{limit link}})$$



$\square$

Let us discard generators which are *composite knots*:



(Similarly, discard composite and split links.)

Thus we consider only <u>prime</u> generators.

genus	1	2	3	4
# prime generators	2	24	4,017	3,414,819

S-Vdovina; *cf.* (3) below: (Exponential) growth rate is ≥ 400! Details...

3. Enumeration of alternating knots by genus

A fundamental tool is the Flyping theorem.

Theorem 8 (Menasco-Thistlethwaite). *Two alternating diagrams of the same (alternating) knot/link are interconvertible by flypes.*

In a generator of k $\sim$-equivalence classes, there are $\sim n^{k-1}$ alternating diagrams of n crossings.

M-T helps taking care of flypes and symmetries, and so:

Theorem 9 (S.). *Let $a_{n,g}$ be the number of alternating knots of genus g and n crossings. Then the sequence $(a_{n,g})_{n=1}^{\infty}$ for fixed g is almost everywhere periodically polynomial (aepp).*

I.e., $\exists\ p_g$ (period), n_g (initial number of exceptions), and polynomials $P_{g,1},\dots,P_{g,p_g} \in \mathbb{Q}[n]$ with

$$a_{n,g} = P_{g,n \bmod p_g}(n) \text{ for } n \geq n_g\,.$$

Another way of writing this:

$$(a_{n,g}) \text{ aepp} \quad \Longleftrightarrow \quad \sum_n a_{n,g} x^n = \frac{R_g(x)}{(x^{p_g}-1)^{d_g}}, \quad R_g \in \mathbb{Q}[x]$$

(where p_g is the period and $d_g = 1 + \max_n \deg P_{g,n}$).

p_g is in general huge, but the leading terms of $P_{g,n}$ have only period 2!

Theorem 10 (S.-Vdovina).

$$[P_{g,\text{even/odd}}]_{\max} = \frac{n^{6g-4}}{(6g-4)!} \cdot \#\{\text{ maximal even/odd generators }\}$$

Definition 11. A class $\mathcal{C} \subset \mathcal{D}$ of knots is *asymptotically dense*, if

$$\lim_{n\to\infty} \frac{\#\mathcal{C} \cap \{c(K) = n\}}{\#\mathcal{D} \cap \{c(K) = n\}} = 1\,.$$

Example 12. An example aside: non-alternating links are asymptotically dense in the class of all links (Thistlethwaite).

Max generators are special alternating, thus:

Proposition 13 (S.-Vdovina). *Among alternating knots of given genus, special alternating ones are asymptotically dense.*

Let

$$C_{g,\text{even/odd}} = \#\{ \text{ maximal genus } g \text{ even/odd generators } \}.$$

A description in [S-V] of maximal generators yields:

Theorem 14 (S-V).

$$400 \le \liminf_{g\to\infty} \sqrt[g]{C_{g,*}} \le \limsup_{g\to\infty} \sqrt[g]{C_{g,*}} \le \frac{2^{20}}{3^6} \approx 1438.38\,. \tag{3}$$

[later (S.) $\limsup \le (2^{130/7}31^{2/7})/3^6 \approx 1425.39$.]

Tool: Wicks forms

4. Wicks forms / Markings

Definition 15. A *maximal Wicks form* w is a cyclic word in the free group over an alphabet with the following 3 conditions:

1) Each letter a and a^{-1} appears exactly once in w.

2) $w \ni$ no subwords of the form $a^{\pm 1}a^{\mp 1}$.

3) (maximality) $a^{\pm 1}b^{\pm 1} \in w$ and $b^{\pm 1}c^{\pm 1} \in w$ (signs independently choosable) $\Rightarrow c^{\pm 1}a^{\pm 1} \in w$ (for proper to be chosen signs).

w, w' *equivalent* up to cyclic permutation and permutation of letters (and inverses)

First studied by Wicks, then Comerford-Edmunds, Culler, Bacher-Vdovina.

(Bacher-Vdovina) duals of 1-vertex triangulations of oriented surfaces:

- number of letters $= 6g - 3$ for some $g > 0$

- label the edges of a $6g - 3$-gon X by letters of w and reverse the orientation induced from the one of X on edges corresponding to inverses of letters.

- identify the edges labelled by each letter and its inverse according to their orientation.

- surface S orientable of genus g. Call g the *genus* of the Wicks form.

- $\partial X \to$ *3-valent (cubic)* graph $G \subset S$ is 1-skeleton of 1-face cell complex (edges of $X \simeq$ letters $a^{\pm 1}$, vertices $\simeq$ triples in maximality property). Dual is 1-vertex triangulation.

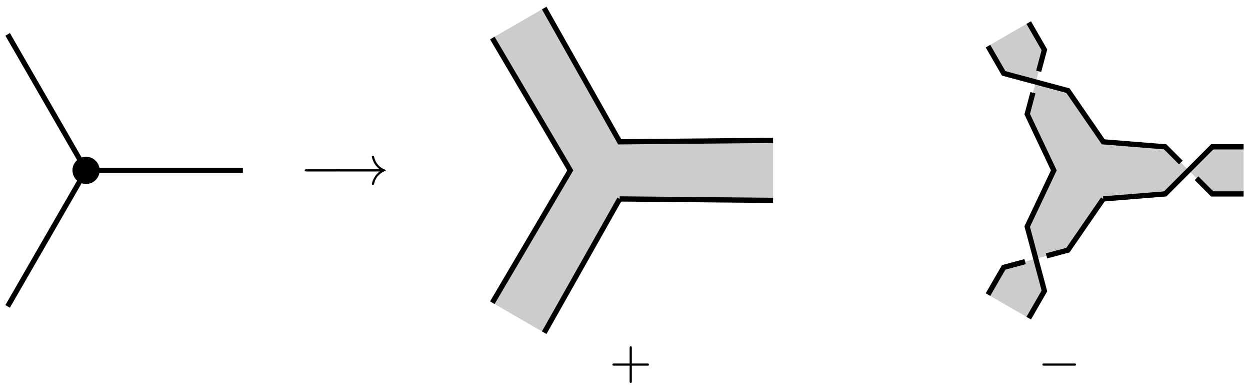
Definition 16 (S-V). *Maximal planar* Wicks form $w :\Longleftrightarrow$ its graph $G \subset S$ planar and *3-connected* (no ≤ 2 edges removed disconnect).

Lemma 17.

$$\left\{ \begin{array}{c} \textit{maximal genus} \\ \textit{g generators} \end{array} \right\} \simeq \left\{ \begin{array}{c} \textit{maximal planar Wicks} \\ \textit{forms of genus } g \end{array} \right\}.$$

G 3-connected $\xrightarrow{\text{Whitney's theorem}}$ $G \subset S^3$ unique. $G \subset S$ is determined by a $+/-$ *marking* of vertices G in planar embedding.

Marking gives Wicks form iff *thickening*

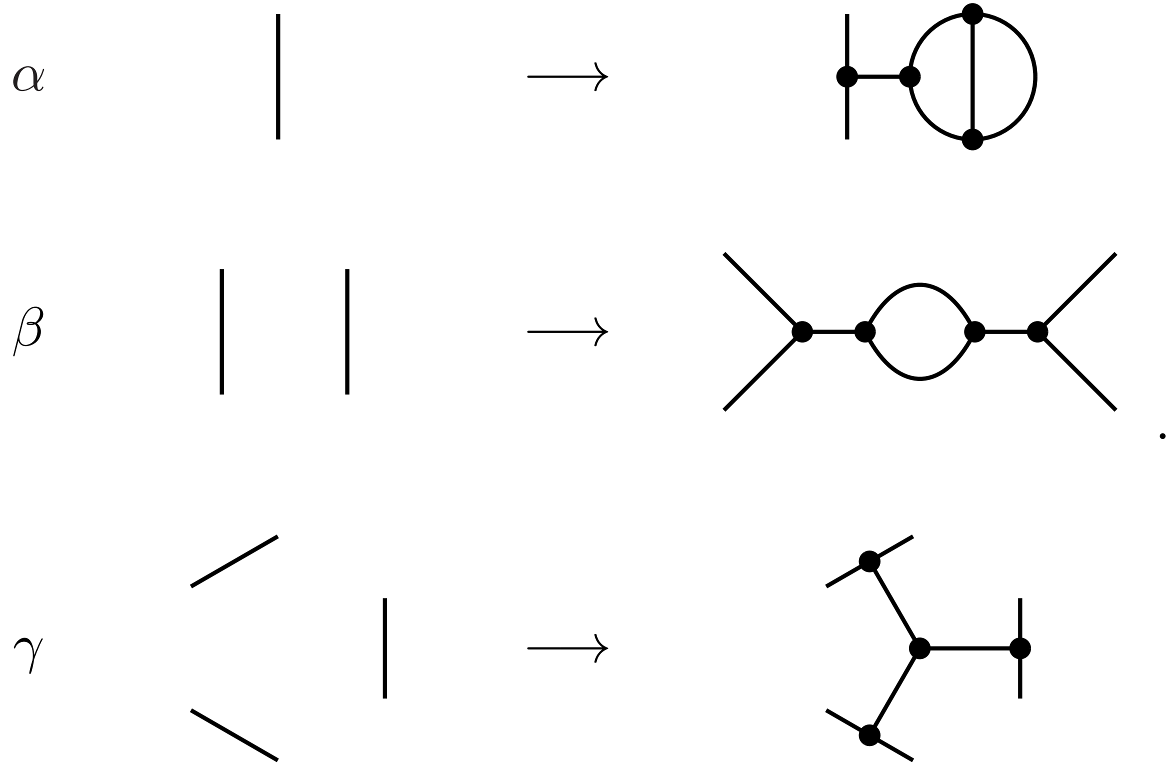


(4)

has one ∂ component (*knot marking*).

Vdovina: 3 elementary operations W forms of genus $g \to$ W forms of genus $g+1$

effect on the graphs: *graphic α, β and γ construction*



(We used γ to prove the left inequality in (3).)

More about this later…

5. Tables of generators

If we sort the 24 prime generators of genus $g = 2$ according to number of crossings and $\sim$-equivalence classes, we obtain the following table:

$\# \sim$ \ c	5	6	7	8	9	10	11	12	13	total
4				1						1
5	1	1	2			1				5
6		1	1	1	3			1		7
7				1	2	1	1			5
8						2	2			4
9								1	1	2
total	1	2	3	3	5	4	3	2	1	**24**

(For $g = 1$ the 'table' is not very revealing.)

The table for $g = 3$:

$\# \sim$ \ c	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total
6						4												4
7	1	2	5	8	11			9										36
8		6	10	21	22	30	44			13								146
9			4	16	42	72	64	55	68			7						328
10				2	15	51	104	159	119	52	45			2				549
11					1	10	49	120	194	211	130	20	14					749
12						1	5	32	112	220	229	154	75	2	1			831
13							1	2	17	63	170	252	178	48	18			749
14									1	4	22	63	132	163	82			467
15											2	3	12	25	47	46	23	158
total	1	8	19	47	91	168	267	377	511	563	598	499	411	240	148	46	23	**4017**

And finally the picture for $g = 4$:

c / $\# \sim$	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	total
8								29	B																	29
9	1 A	2	10	28	71	104	147			145																508
10		21	72	210	356	557	660	819	1092			369														4156
11			48	257	766	1791	2942	3832	3080	2804	3188			447												19155
12				55	487	2033	4734	8585	12145	13523	8500	5168	4707			313										60250
13					56	548	3087	9661	19112	27552	31293	27717	14629	5427	3876			111								143069
14						46	590	3519	13251	32388	52870	61747	53398	35540	15787	3173	1827			20						274156
15							41	489	3584	14749	41049	78373	102880	95709	61646	28311	10626	965	465			1				438888
16								27	356	2814	13781	42566	90877	135278	138221	100392	48096	13094	4195	115	49					589861
17									14	231	1854	9704	34955	83859	141210	163710	125842	68515	24978	2942	837					658651
18										4	96	989	5258	20307	56939	110240	150023	136642	75688	27646	8127	172	46			592177
19											4	25	300	2109	8414	25220	57598	92985	105424	77316	29771	5059	1302			405527
20													6	52	401	2181	6905	17039	32977	45891	44939	27879	7828	E	F	186098
21													C	D	9	36	205	876	2328	4882	8272	10236	9024	5094	1332	42294
total	1	23	130	550	1736	5079	12201	26961	52634	94210	152635	226658	307010	378728	426503	433576	401122	330227	246055	158812	91995	43347	18200	5094	1332	**3414819**

This already displays typical features:

- Entries lies in the angle between $\{c = \# \sim\}$ (left) and $\{c = 2\# \sim\}$ (right critical line).

- Entry A: only generator in first column ($c = 2g + 1$) the $(2, 2g + 1)$-torus knot, giving the series of odd pretzel knots.

- Entry B: only non-zero entry in first row ($\# \sim = 2g$). Generators look like a disk with $2g$ Hopf bands plumbed. [explain how and linking graph]

  For genus $g = 2$ gives, and for $g > 2$ includes the 2-bridge knot generator (where linking graph is a tree).

- Entry C: maximal generators on left critical line ($c = \# \sim = 6g - 3$). Correspond to planar bipartite 3-connected cubic graphs with odd number of spanning trees. $\exists$ for $g = 1$ (θ-curve) and $g \geq 6$, but not for $g = 2, \ldots, 5$.

  [Aside: let G be planar bipartite graph, and $t(G)$ the number of spanning trees of G. Then:

$$\begin{aligned} t(G) \equiv 3 \bmod 4 \quad &\Longrightarrow \quad \chi(G) \equiv 3 \bmod 4 \\ t(G) \equiv 1 \bmod 4 \quad &\Longrightarrow \quad \chi(G) \equiv 1 \bmod 4 \\ t(G) \equiv 2 \bmod 4 \quad &\Longrightarrow \quad \chi(G) \equiv 0 \bmod 2 \end{aligned}$$

The only proof I know uses knot theory. A graph-theoretic proof?]

- Entry D: maximal generators with $c = 6g - 2$ crossings. Always zero; even for links. ($\nexists$ triangulation of the square with even valence vertices.)
- Entries E and F: final two columns. These are $c = 10g - 7$ and $10g - 8$ (for $g > 1$). Non-zero (V. examples), and only non-zero entries in their column: all generators are maximal for $10g-8$ (when $g > 2$) and $10g-7$ crossings (for $g > 1$).

How do I obtain these tables? This is far from routine. Every new genus requires an entirely different idea!

- $g = 1$ is easy and 'folklore' (apart from Brittenham, observed also by Rudolph)
- $g = 2$ using a check in the knot tables
- $g = 3$ using Wicks forms: in Bacher-Vdovina's list of maximal genus 3 Wicks forms, replace each letter by 0, 1 or 2 unlinked crossings and test realizability. Took 3–4 days (on computer).

- $g = 4$ using the (reverse) Hirasawa algorithm. 8 minutes for $g = 3$, and $1\frac{1}{2}$ months for $g = 4$.
- $g = 5$ is (probably forever?) hopeless

[explain Hirasawa algorithm]

Fact 18.

$$\textit{every series} \overset{\substack{\text{Hirasawa}\\\text{algorithm}}}{\subset} \textit{a special series} \overset{\text{S-V}}{\subset} \textit{maximal (generator) series},$$

and Seifert graph of maximal series is (2-)3-valent.

Note: if D is in the series with Seifert graph G, then $\#\sim(D) = \#$ edges in G after removing val-2 vertices

This way I was able to obtain also the <u>bottom rows</u> for $g = 5, 6$.

Method: `plantri` of Brinkmann and McKay + symmetry groups calculated by MATHEMATICA™.

Small application.

$$\lim_{n\to\infty} \frac{a_{2n\pm1,g}}{a_{2n,g}} = \frac{\#\text{ max odd generators}}{\#\text{ max even generators}}$$

evaluates as follows:

genus	1	2	3	4	5	6
# m.e. generators	0	1	74	21124	8307392	3971937256
# m.o. generators	1	1	84	21170	8310928	3971965116
$\frac{\text{odd}}{\text{even}} \approx$	∞	1	1.3514	1.00218	1.00043	1.00001

Combinatorialists know enough to ascertain:

Theorem 19.

$$\frac{\#\ \textit{maximal odd gens of genus } g}{\#\ \textit{maximal even gens of genus } g} \quad \xrightarrow[g\,\to\,\infty]{(\textit{exp. fast})} \quad 1\,.$$

Something we don't know is:

Conjecture 20.

$$\#\text{ max odd gens of genus } g > \#\text{ max even gens of genus } g \quad (\text{for } g > 1)$$

There is an explanation of both statements from the B-V work (later).

6. Further applications of generators

[only keywordwise, since would get too long]

- enumeration problems (already discussed)
- hyperbolic volume (more below)
- signature of positive (and k-almost positive) links

- exactness of the Morton-Williams-Franks braid index inequality and existence of minimal string Bennequin surfaces for alternating knots up to genus 4

- conjectures of Hoste and Fox (Trapezoidal conjecture) on Alexander polynomial of alternating knots (up to genus 4)

- Thurston-Bennequin invariant for Legendrian (and transverse) links

- non-triviality of the Jones ($k \leq 3$) and skein (HOMFLY-PT) polynomial ($k \leq 4$; in special diagrams $k = 5$) of k-almost positive knots

- examples of unsharp Morton inequality for canonical genus

- crossing number estimates for semiadequate links

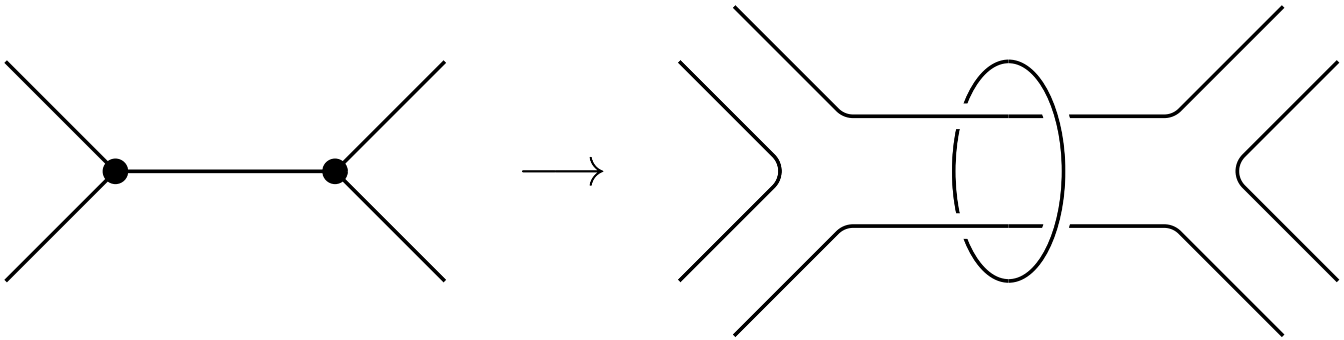
- wave move unknotting conjecture and number of Reidemeister moves needed for unknotting

- hyperbolicity of the canonical genus two knots

7. Back to hyperbolic volume

Recall we were interested in v_χ from (2).

3-conn. 3-valent planar graph $G \to$ *unoriented* link L_G:

 . (5)

Fact 18 + Thurston's hyperbolic surgery theorem $\Rightarrow$

Corollary 21.

$$v_\chi := \max_{\chi(G)=\chi} \mathrm{vol}(L_G)\,, \tag{6}$$

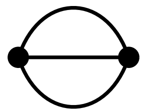
maximum taken over 3*-connected* 3*-valent planar graphs. (And the supremum* v_χ *is attained by special alternating links.)*

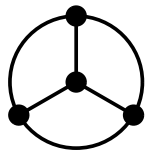
So the question is now: what is $\mathrm{vol}(L_G)$?

For the following let

$$\begin{aligned} V_0 &\approx 1.01494\,, \quad \text{volume(regular ideal tetrahedron)}\,, \\ V_8 &\approx 3.66386\,, \quad \text{volume(regular ideal octahedron)}\,. \end{aligned}$$

and

$\theta \;=$  the theta-curve (7)

$\tau \;=$  the tetrahedral graph (8)

Estimates.

- ‘easy’ observation (Brittenham ’98): $\mathrm{vol}(L) \leq 4V_0 c(L)$ ($c(L)$ crossing number) $\Rightarrow \mathrm{vol}(L_G) \leq 16V_0 \# \sim(D)$;

- (Lackenby '04) $\mathrm{vol}(L_G) \leq 16V_0(\# \sim(D) - 1)$ [in fact works for *twist number* $t(D) \leq \# \sim(D)$] $\Rightarrow$

$$v_\chi \leq 16V_0(-3\chi - 1)$$

- Agol and D. Thurston (appendix to Lackenby's paper) $\Rightarrow$ factor $16 \to 10$ (and asymptotically sharp, but not for fixed χ)
- better approach: v.d. Veen '08 (+C.Adams '85, C.Atkinson '09); below

Disclaimer! The following pictures are taken from R. v.d. Veen, *The volume conjecture for augmented knotted trivalent graphs*, Algebr. Geom. Topol. **9(2)** (2009), 691–722.

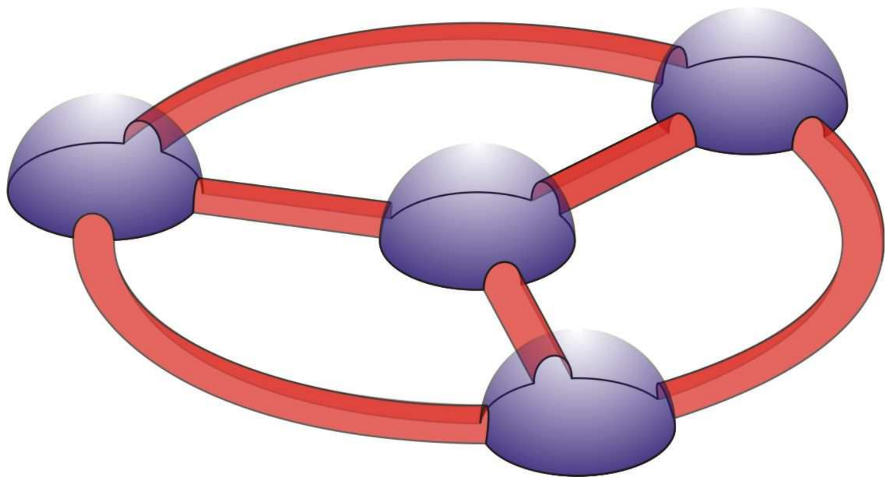
[now we explain a bit of v.d.Veen's work]

$G \subset S^3$ graph; $N(G)$ neighborhood of G. Want hyp. structure on $S^3 \setminus N(G)$.

If all $\partial N(G)$ is cusp, structure is independent under vertex slide and not interesting (in particular no planar G is hyp.)

Thus different structure:

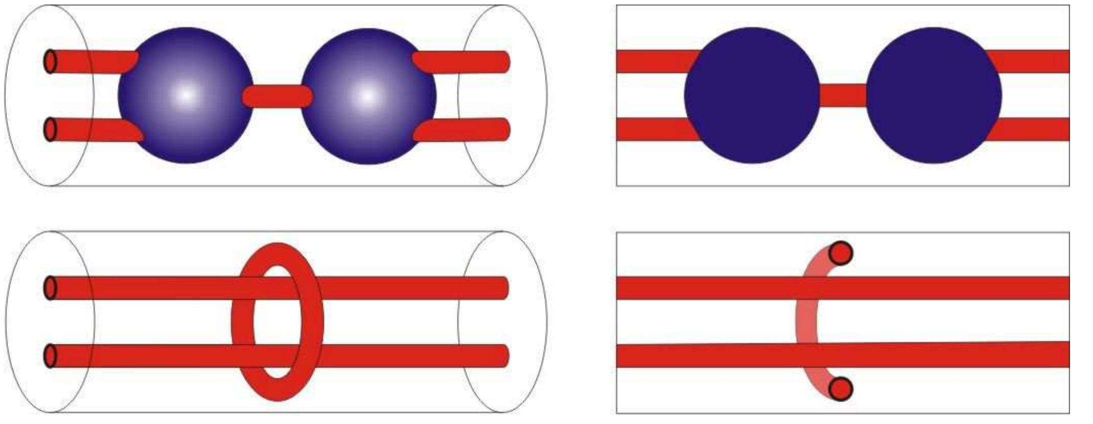
- each vertex of G: geodesic 2-sphere
- each edge of G: geodesic cylinder $\rightarrow$ cusps (later link components)



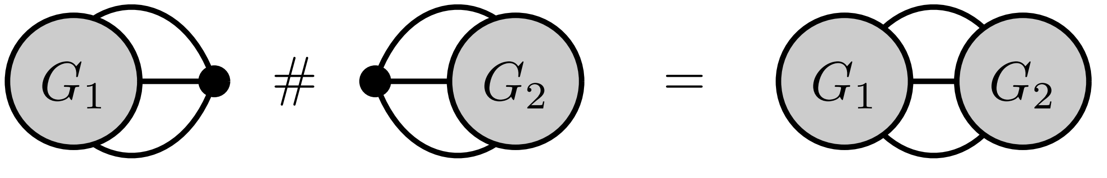
<u>If</u> $S^3 \setminus N(G)$ is hyp. (and $G \neq \theta$ 3-connected planar always ok; see below), some sort of (Mostov) rigidity holds $\Rightarrow \exists$ volume; *graph volume* $\mathrm{vol}(G)$ of G.

Definition 22. *vdV's moves* (which turn G into a link):

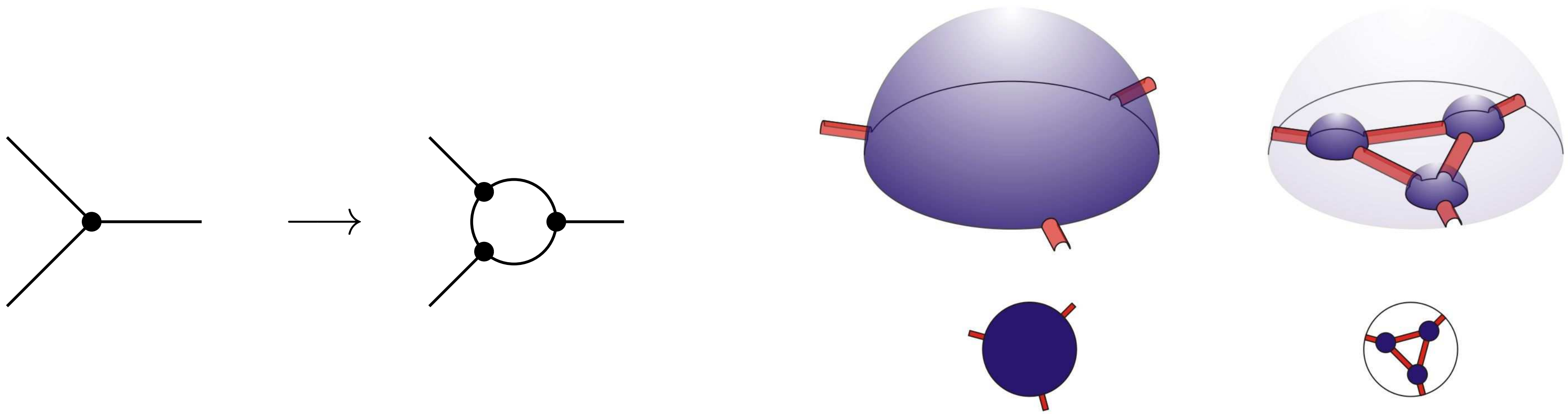
- *(un)zipping* (almost same as (5)) preserves volume



- *composition* (essentially) '#' adds volumes



When $G_2 = \tau$, *triangle* $\triangle$ *move* (vdV considers only this case):



Note: despite that # is highly ambiguous, additivity

$$\mathrm{vol}(G \# G') \;=\; \mathrm{vol}(G) + \mathrm{vol}(G')$$

holds for all possible ways of doing '#'!

$G(\neq \theta)$ planar $\Rightarrow$ $S^3 \setminus N(G)$ = 2 (equal) ideal $\pi/2$-angled polyhedra with 1-skeleton = *median graph* of G (realizable by Andreev's theorem when G 3-conn.) $\Rightarrow$ G hyperbolic.

Zipping $\Rightarrow$

$$\mathrm{vol}(L_G) = \mathrm{vol}(G'),$$

where G' is obtained from G by $\triangle$ move at each vertex $\Rightarrow$

$$\mathrm{vol}(L_G) = \mathrm{vol}(G) + 2V_8 \cdot v(G),$$

with $v(G) := \#\{ \text{ vertices of } G \}$.

By using Atkinson's estimate on polyhedral volume:

Proposition 23.

$$V_8\,(-6\chi - 2) \le v_\chi \le V_8(-7\chi - 1)\,. \tag{9}$$

Remark 24. Lower bound attained when we glue only octahedra; for $G = \tau\#\tau\#\ldots\#\tau$. VdV proves Volume conjecture for (unzippings of) such G.

More generally, iterated composition shows:

Corollary 25. $\exists$ *'stable volume-χ ratio'*

$$\delta = \lim_{\chi\to-\infty} \frac{v_\chi}{(-2-6\chi)V_8} = \sup_{\chi<0} \frac{v_\chi}{(-2-6\chi)V_8},$$

and

$$1 \leq \delta \leq \frac{7}{6} . \tag{10}$$

Computation (below) improves lower bound to $\approx 1.08796 \Rightarrow$ upper bound sharp up to $< 10\%$.

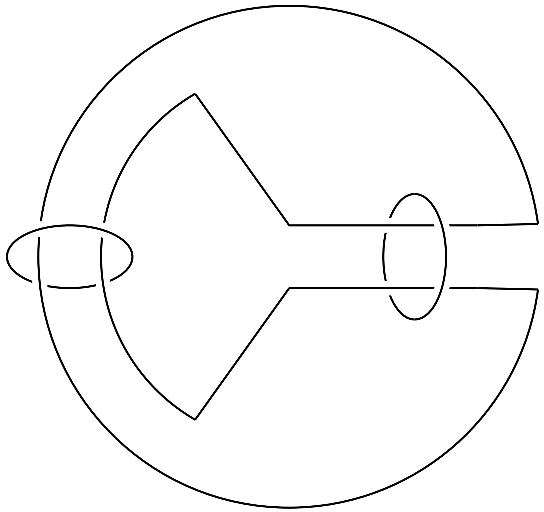
Computation.

first simplification

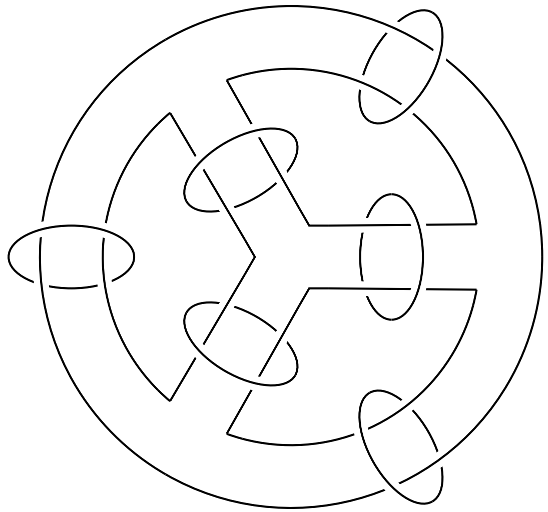
instead of $\mathrm{vol}(L_G)$ can calculate $\mathrm{vol}(L'_G)$: unzip G along a *perfect matching*.

Definition 26. *Perfect matching* $S \subset \{$ edges of G $\}$ s.t. $\forall$ vertex of G $!\exists$ incident edge $\in S$.

L'_G has (at least) 3 times fewer crossings than L_G! (Faster to obtain vol with J.Weeks' SnapPea)



L'_τ



L_τ

Remark 27. Always $\exists$ perfect matching (in fact, $\exists$ exponentially many, a 40-year-old problem solved recently Esperet-Kardoš-King-Král'-Norine '11).

second simplification

composition $\Rightarrow$ enough to consider *cyclically 4-connected (c4c)* G.

Definition 28. Cubic G *cyclically 4-connected* $:\Leftrightarrow$ ≤ 3 edges disconnecting G are 3 edges incident to a vertex of G (in particular, 3-conn.)

Or: $G = G_1 \# G_2 \Rightarrow G_1 = \theta$ or $G_2 = \theta$

Result:

χ	# c4c G	max. vol. $v_\chi \approx$	$\dfrac{v_\chi}{(-2-6\chi)V_8} \approx$
-1	0	$4V_8$	1
-2	1	$10V_8$	1
-3	0	$16V_8$	1
-4	1	82.7139821	1.02616
-5	1	105.8287878	1.03159
-6	2	129.3489143	1.03835
-7	4	153.3818722	1.04659
		$\dots$	
-19	136610879	444.7966230	1.08394
-20	765598927	469.2471319	1.08538
-21	4332047595	493.7021266(?)	1.08669(?)
-22	?	518.1952668?	1.08796?

(Last row uses only – heuristically – cycl. 5-conn. graphs.)

Conjecture 29. Right ratio always rising, i.e., maximal vol(G) attained for c4c G. (In fact, can conjecture c5c for $\chi \leq -12$.)

8. The relation between volume and the sl_N polynomial

We studied maximal volume v_χ for links: if $\chi_c(L) = \chi$, then L has $n = 2-\chi, \chi, \dots, \chi \bmod 2 (\in \{1,2\})$ components.

How about fixing n (in particular knots $n = 1$)?

Definition 30. Recall marking $O : \{$ vertices of G $\} \to \{+, -\}$ and thickening (4) we call $S = S_{G,O}$.

Let $L_{G,O} = \partial S_{G,O}$ and $n_{G,O} = n(L_{G,O})$ be # of components.

Again for $\chi(G) = \chi$, have $n = n_{G,O} = 2 - \chi, \chi, \dots, \chi \bmod 2$ components. Call O

- *even/odd* if *diagram of* $L_{G,O}$ has even/odd crossings,
- *minimal* if $L_{G,O}$ has $n(L_{G,O}) = 1$ or 2 components, and
- *spherical* if $L_{G,O}$ has $n = 2 - \chi$.

[starting again a (last) detour]

Thickenings occur in the calculation of the sl_N *polynomial*.

Theory of Vassiliev invariants (Kontsevich, Bar-Natan): for a (semisimple) Lie algebra $\mathfrak{g}$ $\exists$ invariant, *weight system*, $W_{\mathfrak{g}}$ of (uni-)trivalent graphs G.

For $\mathfrak{g} = sl_N$ the polynomial $W_{sl_N}(G) = W_N(G)$ can be calculated by:

$$W_N(G) = W_{N,+}(G) - W_{N,-}(G)\,,$$

with

$$W_{N,+/-}(G) = \sum_{O \text{ even/odd}} N^{n_{G,O}}\,.$$

$W_N(G) \in \mathbb{Z}[N]$; only even or odd degree terms, maximal possible degree $2-\chi$.

Theorem 31 (Bar-Natan's version of Four color theorem 4CT).

$$\begin{array}{ccccc} \#\{\ \textit{spherical markings of } G\ \} \neq 0 & \Rightarrow & \#\{\ \textit{four-colorings of } G\ \} \neq 0 \\ \| & & \wr\| \ \textit{(if G planar)} \\ [W_N(G)]_{2-\chi} \qquad \neq 0 & \Rightarrow & W_2(G) \qquad \neq 0 \end{array}$$

Little else known on $W_N(G)$.

Easy:

$$W_N(\theta) \;=\; 2N\,(N^2-1) \mid W_N(G)\,, \quad \text{and} \quad W'_N(G) := \frac{W_N(G)}{W_N(\theta)}$$

has only even/odd powers.

Lemma 32. $W'_N(G_1 \# G_2) = W'_N(G_1) W'_N(G_2)$ *(for <u>all</u> '#').*

Remark 33. Vogel '96 introduced an algebra Λ and proved something similar (under restrictions) for arbitrary Lie (super) algebra (showing $\exists$ non-Lie algebraic weight systems).

Recall that degree 1 (=knot markings) studied by B-V.

Definition 34 (following B-V). Call a vertex of G *good/bad* in a marking O if changing marking of v in O preserves/changes $n_{G,O}$.

Lemma 35 (B-V). *A knot marking O for G with $\chi(G) = 1 - 2g$ has $2g$ bad and $2g - 2$ good vertices.*

($\Rightarrow$ number of good/bad vertices independent of O and depends on G only via $\chi(G)$!)

Corollary 36. *If $G \neq \theta$ (even non-planar),*

$$[W_{N,+}(G)]_1 = [W_{N,-}(G)]_1 \quad \Rightarrow \quad [W_N(G)]_1 = 0\,.$$

Modding out by symmetries of G returns us to maximal generators:

- It is *known* that symmetries fade (fast) when $g \to \infty \Rightarrow$ theorem 19.
- Even markings still *seem* to have a few more symmetries than odd markings $\Rightarrow$ conjecture 20.

Remark 37. Nothing similar to lemma 35 true for $n > 1$. Of course, by lemma 32: $G = G_1 \# \ldots \# G_n \Rightarrow$ often low degree terms vanish, but many c4c G have degree 2 or 3 terms.

But there is a criterion for good vertices in the general case:

Theorem 38. $\exists$ *good vertex of* $(G, O) \iff$ *a component of diagram of* $L_{G,O}$ *has a self-crossing.*

If G 2-connected, Whitney's theorem $\Rightarrow$ spherical markings are even.

Corollary 39. *G 2-connected (cubic), O non-spherical marking $\Rightarrow \exists\ O'$ of opposite parity with $n_{G,O} = n_{G,O'}$.*

(In some sense a counterpart to Whitney's theorem on higher genus surfaces.)

Relation to volume.

Computations led to the following (*a priori* naive) question:

Question 40. Is there a relation between $W_N(G)$ and $\mathrm{vol}(G)$? (Bluntly): does one determine other?

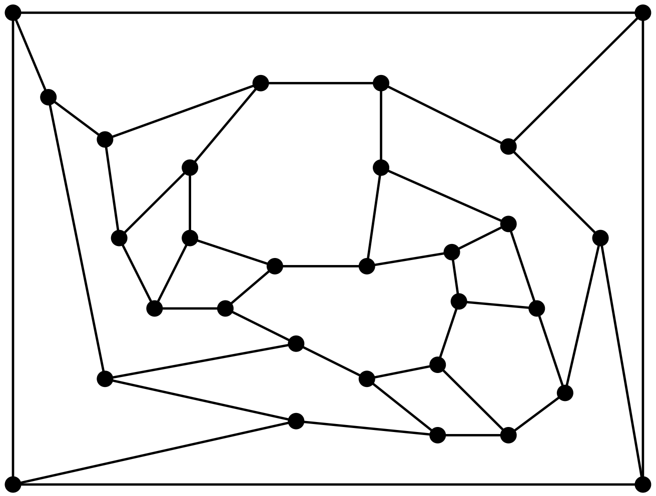
Analogy: both $W_N(G)$ and $\mathrm{vol}(G)$ behave 'well' under composition

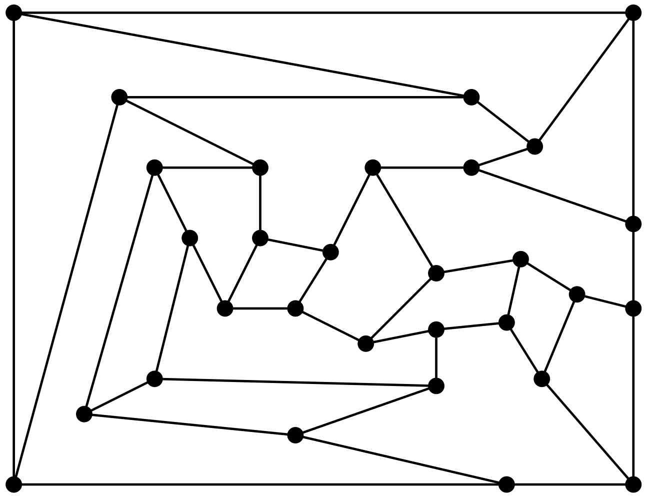
$\Rightarrow$ look at c4c G.

Result:

$\mathrm{vol}(G) \not\Rightarrow W_N(G)$	G_1, G_2 of $\chi = -9$ equal vol, different W_N
$\mathrm{vol}(G) \not\Leftarrow W_N(G)$	counterexamples only from $\chi = -15$ (3-4 months of computation and skillful programming needed!)

One pair:





$\mathrm{vol}(G) \approx 120.7043\underline{405}$ $\qquad$ $\mathrm{vol}(G') \approx 120.7043\underline{733}$

$$W_N(G) = W_N(G') = 10496N^3 - 1536N^5 - 9760N^7 - 7100N^9 + 6672N^{11} + 1156N^{13} + 70N^{15} + 2N^{17}$$

Remark 41. It is not that $W_N(G) = W_N(G')$ occurs seldom. E.g., in $\chi = -12$ $\exists$ 1357 c4c graphs, 617 coincidences of W_N (i.e., 740 diff. values)

In 10^5's of computed c4c examples (til $\chi = -17$, $\approx 55\%$ of $\chi = -18$), $W_N(G) = W_N(G') \Rightarrow \mathrm{vol}(G) = \mathrm{vol}(G')$, and even in discrepancies W_N predicts vol with unusual accuracy:

$$\frac{\big|\,\mathrm{vol}(G) - \mathrm{vol}(G')\,\big|}{\mathrm{vol}(G)} < 6 \cdot 10^{-7} \,.$$

So what is mystery around question 40? Why answer so little naive?

9. Independence of volume on number of components

[end of the last detour]

Recall: how about maximal volume of *knots* (or fixed number of link components)?

$$v_{n,\chi} \;=\; \sup\{\,\mathrm{vol}(L) \,:\, \chi_c(L) = \chi,\; n(L) = n\,\} \tag{11}$$

New version of (6),

$$v_{n,\chi} \;:=\; \max_{\chi(G)=\chi} \mathrm{vol}(L_G)\,,$$

changes by taking only G with n-component markings.

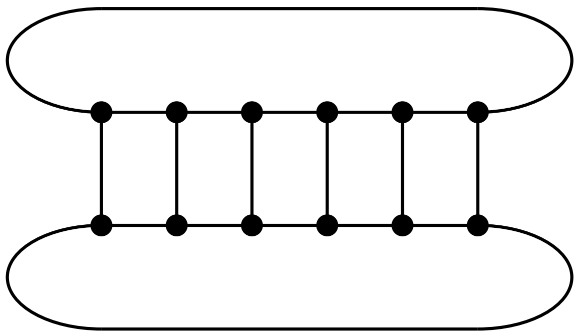
Thus: what (3-conn. cubic planar) G have n-component markings?

Lemma 42. *G has n-component non-spherical marking $\Rightarrow \exists$ $n+2$-component.*

Proof (easy). Take n-component marking O and successively make all signs $+$ (or $-$) to get spherical marking. $\square$

So question is: given G what is *minimal* component marking, $\min \deg_N W_{N,\pm}$?

Example 43. Can be arbitrarily large! Consider the 'ladder' A_n of n stairs (or spokes); for $n = 6$



.

$A_n \to B_n$ by $\triangle$ move $(\ldots \#\tau)$ at each vertex. Then

$$\min\deg_N W_{N,\pm}(B_n) \geq \frac{1}{6} v(B_n) \to \infty \,.$$

(Question: 'worst' case?)

When $\exists$ minimal marking ($\min\deg \leq 2$)? Maybe such examples arise because we build $\#$ in the 'wrong' way?

Lemma 44. *G_1, G_2 planar 3-conn. cubic graphs w/ minimal marking $\Rightarrow$ $\exists$ a way to do $G_1 \# G_2$ s.t. it has minimal marking.*

(Proof uses again B-V lemma 35.)

So question reduces to c4c graphs. Also $n = 2$ reduces easily $n = 1$, and we are back to knots. Of course,

knots $\Rightarrow$ knot markings $\Rightarrow$ Wicks forms $\Rightarrow$ Vdovina constructions α, β, γ

give *exact* description, but it's recursive and takes no account of c4c.

So I crunched a bit until I had what I wanted:

Theorem 45. *Every cyclically 4-connected planar 3-valent graph of odd χ has a knot marking.*

. . . with the intended application:

Corollary 46. $v_{n,\chi} = v_\chi$, *i.e., maximal volume independent of number of components.*

Proof. Take G with maximal $\mathrm{vol}(L_G)$ (for given χ), decompose it into c4c pieces, (get a minimal marking on each c4c piece,) and put the pieces correctly together. $\square$

Remark 47. If conjecture 29 is true, then this whole 'rearrangement' is unnecessary.

Thank you!

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