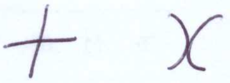
Introduction to knot computing

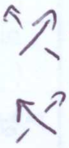
① knot $S^1 \hookrightarrow S^3$ smooth

link $S^1 \cup \ldots \cup S^1 \hookrightarrow S^3$ diagram planar curve(s)

with transverse xs

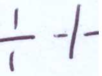

← not

orientation

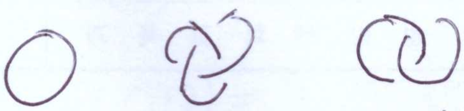

+crossing
−crossing

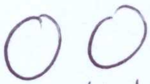
not


+ crossing information



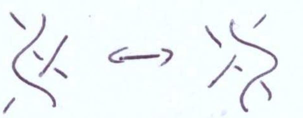
mirroring

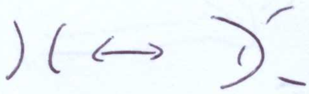

unknot trefoil Hopf link

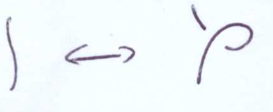

unlink

equivalence of links

$$\underset{\text{link}}{D_1} \sim \underset{\text{link}}{D_2} \iff \exists \text{ sequence of Reidemeister moves}$$


RIII


RII


RI

problem R moves can raise number of crossings

(+mirror image)

"local moves"

choose something outside

$D_1 \to D_1' \to \ldots \to D_2$

how long?

Hass-Lagarias $\underset{\text{unknot}}{O} \leftarrow \cdots\cdots \leftarrow D$ (n crossing) Hayashi $OO\ldots O$ trivial link

length $\leq 10^{10^7 n}$ (practically useless)

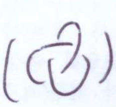
<u>Def:</u> K knot (L link)

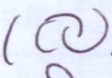
crossing number $= \min\{$ crossing # of diagram D : D diagram of K $\}$

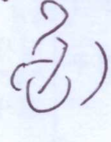
$c(K)$ $\qquad c(D)$

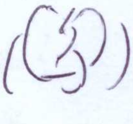
ex

$c(O) = 0$

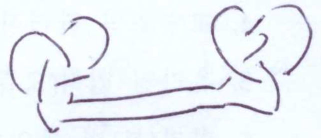
$c($  $) = 3$

$c($  $) = 2$ figure-8

$c($  $) = 3$

$c($  $) =$

<u>Def:</u> connected sum

$K_1 \# K_2$  #  

exercise $K \# O = K$

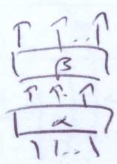
<u>open problem:</u> $c(K_1 \# K_2) = c(K_1) + c(K_2)$?

② braid

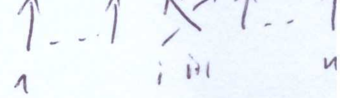
$\sigma_1 \sigma_2^{-1} \sigma_3^{2} \sigma_1 \sigma_2^{-1}$ $\quad 1\ 2\ 3\ 4 \quad n=4$

form group by stacking

$n > 1$

$\alpha \cdot \beta =$ 

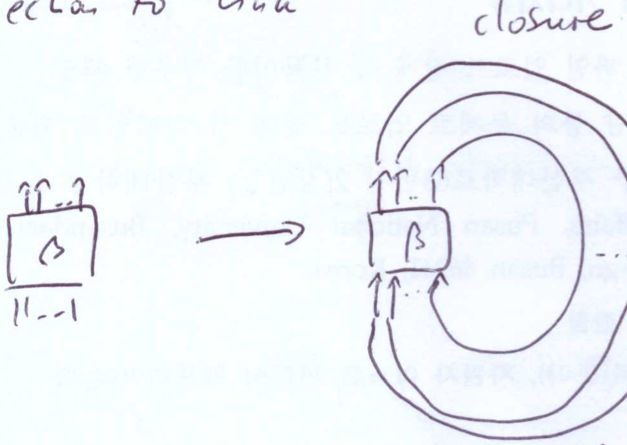
$\sigma_i =$ 

$\sigma_i^{-1} =$ 

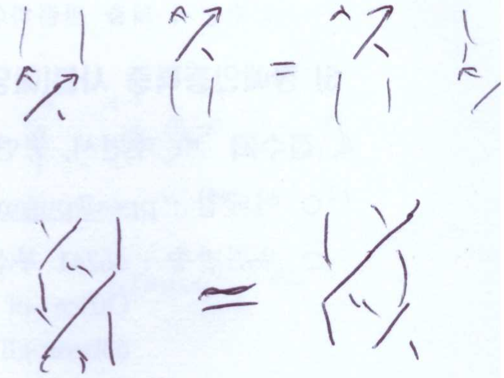
$$B_n = \left\langle \sigma_1 \dots \sigma_{n-1} \;\middle|\; \begin{matrix} \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j|>1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \end{matrix} \right\rangle$$

generators   relations

connection to link   closure



braid   link (oriented)



$b(L) = \min\{n : \exists \beta \in B_n,\ \hat\beta = L\}$

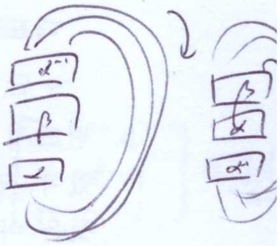
braid index

Th. (Alexander) $\forall$ link L $\exists \beta$: $\hat\beta = L$

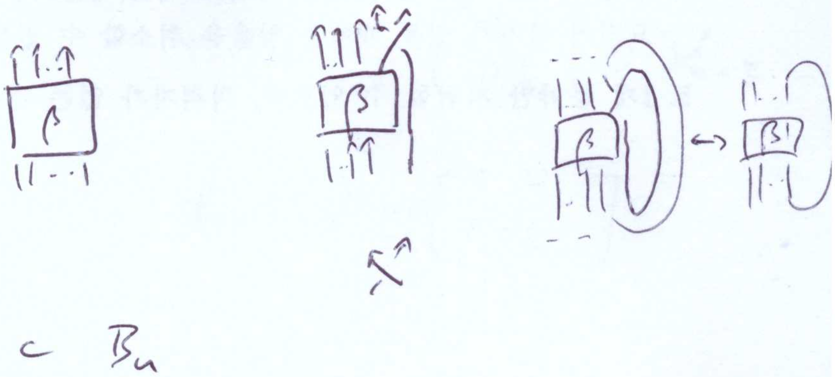
Th. (Markov) $\hat\beta_1 = \hat\beta_2 \iff \beta_1 \to \dots \to \beta_2$ by a sequence of Markov moves

move

M1 conjugacy $\beta \in B_n$, $\alpha \in B_n$ $\beta \to \alpha\beta\alpha^{-1}$



M2 (de)stabilization $\beta \in B_{n-1} \underset{\text{destab}}{\overset{\text{stab}}{\longleftrightarrow}} \beta\sigma_{n-1}^{\pm} \in B_n$

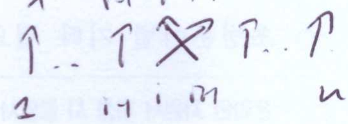


$\beta \in B_{n-1} \subset B_n$

$$\begin{array}{ccc} K_1 = \hat{\beta}_1 & & \beta_1 \\ \vdots & & \downarrow \\ \vdots & & \vdots\ ? \\ \downarrow & & \downarrow \\ K_2 = \hat{\beta}_2 & & \beta_2 \end{array}$$

conjugacy is solved
but stabilization
makes things "complicated"

homomorphisms

$B_n \longrightarrow S_n$ symmetric group $\qquad \sigma_i \longmapsto$ 

permutation

$(B_n, \cdot) \longrightarrow (\mathbb{Z}, +) \qquad \sigma_i \longmapsto 1$ "exponent sum"

$\sigma_1 \sigma_2^{-3} \sigma_3 \longmapsto 1 - 3 + 1 = -1$

$B_4 \longrightarrow B_3 \qquad \sigma_1 \longmapsto \sigma_1 \quad \sigma_3 \longmapsto \sigma_1$
$\qquad \sigma_2 \longmapsto \sigma_2$

Laurent polynomials

$\downarrow \qquad -3t^{-2} + t^{-1} + 5t - \dots$

<u>Burau representation</u>

$$B_n \longrightarrow GL(n-1, \mathbb{Z}[t, t^{-1}])$$

$$M \oplus N = \left[\begin{array}{c|c} M & 0 \\ \hline 0 & N \end{array}\right]$$

$$\psi_{n-1}(\sigma_i) = \mathrm{Id}_{i-2} \oplus \begin{bmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{bmatrix} \oplus \mathrm{Id}_{n-\dots}$$

$$\psi_{n-1}(\sigma_1) = \begin{bmatrix} -t & 1 \\ 0 & 1 \end{bmatrix} \oplus \mathrm{Id}_{n-3}$$

$$\psi_{n-1}(\sigma_{n-1}) = \mathrm{Id}_{n-3} \oplus \begin{bmatrix} 1 & 0 \\ t & -t \end{bmatrix}$$