GRID DIAGRAMS, LINK INDICES, AND STRONG QUASIPOSITIVITY

Hwa Jeong Lee(†), Alexander Stoimenov

Dongguk University, WISE campus,
Department of Mathematics Education,
123, Dongdae-ro,
38066 Gyeongju-si, Republic of Korea
e-mail: `stoimeno@stoimenov.net`,
WWW: `http://stoimenov.net/stoimeno/homepage/`

Gyo Taek Jin

Department of Mathematical Sciences
Korea Advanced Institute of Science and Technology,
34141 Daejeon, Republic of Korea
e-mail: `trefoil@kaist.ac.kr`,

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Abstract. We present a viewpoint on Euler characteristic 0 braided surfaces as grid diagrams. This leads to some results regarding estimates of Thurston-Bennequin invariants of knots, strong quasipositivity of Whitehead doubles, jump numbers of slice-torus invariants, and arc and braid index. We also consider the crossing number of grid diagrams and rectilinear (planar grid) polygons, and versions of the braid index related to braided and strongly quasipositive surfaces.

Keywords: arc index, braid index, rectilinear polygon, Thurston-Bennequin invariant, strongly quasi-positive, Whitehead double, slice-torus invariant.

2020 AMS subject classification: 57K10 (primary), 57M50, 20F36, 05C10, 51E12, 57K33 (secondary)

1 Introduction

This investigation resulted from attempts to understand braided surfaces, in particular Bennequin and strongly quasipositive surfaces. Similar to the case of canonical surfaces [St], we were trying to develop some structural properties. As it turned out, even in the simplest case of Euler characteristic 0, the answer is revealingly complicated, in that these surfaces are essentially equivalent to integer-labeled grid diagrams D for knots. However, despite protruding such complexity, this connection leads to some new viewpoints, and assimilates a number of known and new results. We present them here as an initiation for further study (see the sequel [JLS], and [MS+]). An outline of the paper is as follows.

After compiling preliminaries in §2, we give some simple but useful observations in §3 on crossing number and writhe of grid diagrams. In particular, we determine the maximal crossing number of a rectilinear polygon of given size exactly in the multi-component case (Lemma 3.1) and nearly exactly in the one-component case (Proposition 3.3 and Computation 3.4). As a consequence we obtain an *upper* bound of the crossing number of a link L in terms of its arc index (see Corollary 3.7).

In §4 we refine the grid-band construction [Nu] to understand that Euler characteristic 0 braided surfaces are essentially grid diagrams D, with a framing attached, which we write as $\lambda(D)$. When the surface is strongly quasipositive, then

$$\lambda(D) = -TB(D) \tag{1.1}$$

is identified, up to sign, with the Thurston-Bennequin invariant of D. We will establish this in Theorem 4.9 from introducing a weight model for the Thurston-Bennequin invariant from a grid diagram (Lemma 4.4). As we explain, in conformance with (1.1), we will usually write $\lambda(K) = -TB(K)$, for the maximal Thurston-Bennequin invariant $TB(K)$ of K.

We give some simple applications, including estimates of $\lambda(K)$ in relation to arc index and bridge number (Theorem 4.8 and Lemma 4.18), and Rudolph's determination of the strong quasipositivity of twisted annuli $A(K,t)$ (Corollary 4.13). This is then extended to twisted positively/negatively clasped Whitehead doubles $W_\pm(K,t)$ and Bing doubles $B(K,t)$ (Corollary 4.15).

In §5 we discuss the braid index $b(K)$ and its variants for Bennequin and strongly quasipositive surfaces, and how the arc index $a(K)$ is fundamentally connected to a braid index $b(A(K,t))$ (see Corollary 5.1 and Conjecture 5.8). As a bi-product, we can reprove (and slightly improve) Ohyama's [Oh] inequality (Corollary 5.5). We also study the framing diagram $\Phi(K)$ of a knot K (Definition 4.12) and its cone structure (Theorem 5.10).

Section §6 deals with the jump function j_v of slice-torus invariants v. After we reproduce the Livingston-Naik [LvN] estimate (Proposition 6.3), we extend it with a condition (6.9) on a knot K that resolves the Bennequin-sharpness problem (2.9) for its Whitehead doubles (Corollary 6.4). We also establish (see Lemma 6.5) that positive knots K satisfy this condition.

In §7 we only briefly motivate a more general theory, of "grid-embedded graphs" for braided surfaces of smaller (i.e., negative) Euler characteristic. It should also be noted from the start that grid diagrams of links can be treated by essentially the same approach, without very major modifications, but for technical reasons we stick mostly to knots. Some study of links has been begun in [MS+].

This is the first part of work divided into three parts for length reasons. The second part [JLS] will deal extensively with applications of the HOMFLY-PT polynomial. The final part [St3], given by the second author, deals with extensions from strong quasipositivity to quasipositivity. Many of the results here serve as a motivation for their later analogues and generalizations. (This will be explicitly indicated wherever relevant.)

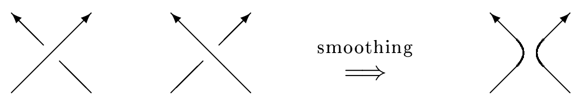
2 Definitions and Preliminaries

2.1 Generalities

We say an inequality '$a \geq b$' is *sharp* or *exact* if $a = b$ and *strict* (or *unsharp*) if $a > b$. We use $\#E$ for the cardinality of a finite set E and $\lfloor x \rfloor$ for 'greatest integer' part of $x \in \mathbb{Q}$. We also afford a few standard abbreviations like 'l.h.s.' (for 'left hand-side'), 'w.r.t.' (for 'with respect to') and 'w.l.o.g.' (for 'without loss of generality').

2.2 Links and genera

All link diagrams and links are assumed oriented. Crossings in an oriented diagram D of a knot K are called as follows.



positive negative smoothed out

(2.1)

The *sign* of a positive/negative crossing is assigned to be ± 1 accordingly. Let $c_\pm(D)$ be the number of positive, respectively negative crossings of a link diagram D, so that the *crossing number* of D is $c(D) = c_+(D) + c_-(D)$ and its *writhe* is $w(D) = c_+(D) - c_-(D)$. We write $s(D)$ for the *number of Seifert circles* of D, which are the circles obtained after smoothing all crossings of D. We write $c(K)$ for the crossing number of a knot K, the minimal crossing number of all diagrams of K. The mirror image of K will be written $!K$, and the mirror image of diagram D (in the form obtained by switching all crossings of D) will be $!D$. If $K = !K$ (up to orientation), we call K *amphicheiral*. We use '$\bigcirc$' to denote the *unknot* (trivial knot) in formulas.

The symbol '#' is used for *connected sum*. (This should be distinguishable from the context from the use of '#' as 'cardinality'.) The number of components of a link L is denoted $\kappa(L)$. The *bridge number* $br(L)$ of L is the minimal number of Morse maxima of L (or equivalently, of any diagram of L). The *(Seifert) genus* $g(L)$ resp. *Euler characteristic* $\chi(L)$ of a knot or link L is said to be the minimal genus resp. maximal Euler characteristic of a *Seifert surface* of L. We have

$$2g(L) = 2 - \kappa(L) - \chi(L)\,.$$

Similarly write $\chi_4(L)$ for the *smooth 4-ball* (maximal) Euler characteristic and

$$2g_4(L) = 2 - \kappa(L) - \chi_4(L)\,.$$

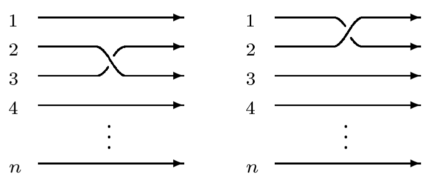
(In the following 4-ball genera and sliceness will always be understood smoothly.) A knot K is *slice* if $g_4(K) = 0$, or equivalently, $\chi_4(K) = 1$. We will refer to the following basic fact: if $\kappa(L) = 2$ and $\chi_4(L) = 2$, then both components of L must be slice (knots), and have linking number 0.

2.3 Braids and braided surfaces

We write B_n for the *braid group* on n *strands* or *strings*. The relations between the *Artin generators* σ_i, $i = 1, \dots, n-1$ are given by

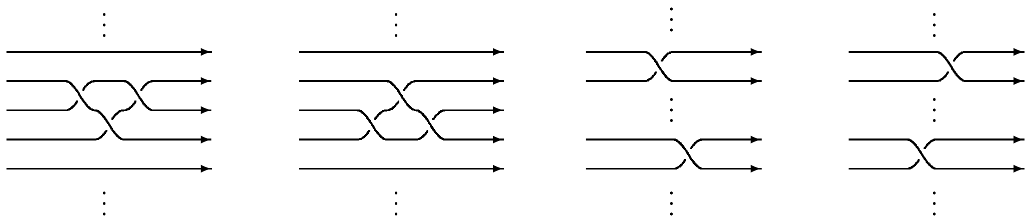
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \le i \le n-2$ and
- $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $1 \le i < j-1 \le n-2$.

In diagrams we will orient braids left to right and number strings $1, \dots, n$ from top to bottom, for example:



σ_2 $\qquad\qquad$ σ_1^{-1}

The relations can then be drawn as follows:



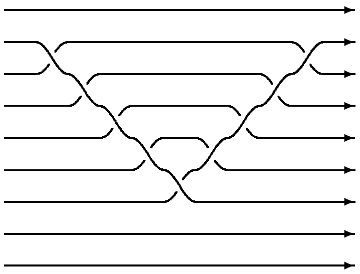
$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ $\qquad\qquad$ $\sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i-j| > 1$

There is a *permutation homomorphism* $\pi : B_n \to S_n$, sending each σ_i to the transposition of i and $i+1$.

We define *band generators* in B_n by

$$\sigma_{i,j} \;=\; \sigma_i \ldots \sigma_{j-2}\sigma_{j-1}\sigma_{j-2}^{-1}\ldots\sigma_i^{-1}\,, \tag{2.2}$$

For example $\sigma_{2,7} \in B_9$ looks



Notice that $\sigma_{i,i+1} = \sigma_i$. A presentation of a braid $\beta \in B_n$ in the form

$$\beta \;=\; \prod_{k=1}^{l} \sigma_{i_k,j_k}^{\pm 1} \tag{2.3}$$

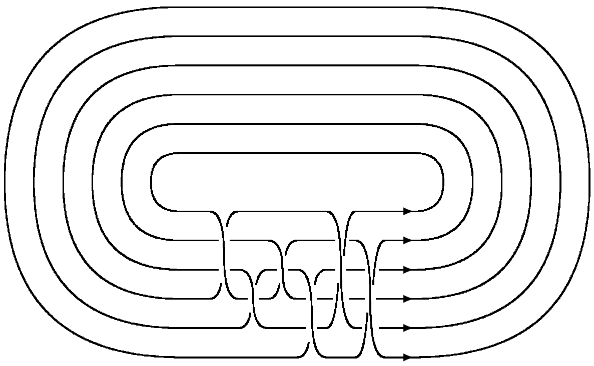
is called a *band presentation*. (See e.g. [BKL].) Usually, it will be more legible to use the symbol

$$[ij] = \sigma_{i,j}$$

when writing band generators in formulas. Similarly we use $-[ij] = \sigma_{i,j}^{-1}$. In certain cases, we even omit the brackets. (See Definition 4.6, where we will extend suitable words in $[ij]$, without negative exponents, also to encode grid diagrams.) Also, when $j = i+1$, we often simply write i for σ_i and $-i$ for σ_i^{-1}, when no ambiguity arises.

The image of β under the abelianization $B_n \to \mathbb{Z}$ is the *writhe* (or exponent sum) of β, and is written $w(\beta)$. Note that $w(\beta)$ can also be calculated as the exponent sum in a band presentation (2.3).

A braid $\beta \in B_n$ whose *closure* $\hat\beta$ is the link L is a *braid representative* of L. Similarly a word for β gives a (braid closure) diagram $D = \hat\beta$ of L. When β is a word, then $w(\hat\beta) = w(\beta)$. A band presentation β naturally spans a Seifert surface of $L = \hat\beta$. Following Rudolph, we call this a *braided surface* of L. For example, $n = 6$ and $l = 6$,



for the 6-braid $\beta = \sigma_{1,4}\sigma_{3,5}\sigma_{2,4}\sigma_{3,6}\sigma_{1,5}\sigma_{2,6}$. The diagram shows the closure $L = \hat\beta$. It is easily seen that the six 'elliptic' disks joined two by two with six twisted bands form a natural Seifert surface of L.

Rudolph [Ru] proves that every Seifert surface is a braided surface. If a braided surface is of minimal genus for L, it is called a *Bennequin surface* of L [BM2].

A link is called *quasipositive* if it is the closure of a braid β of the form

$$\beta = \prod_{k=1}^{\mu} w_k \sigma_{i_k} w_k^{-1} \tag{2.4}$$

where w_k is any braid word and σ_{i_k} is a (positive) standard Artin generator of the braid group. (In [Ru4] there is some overview of this topic.) If the words $w_k \sigma_{i_k} w_k^{-1}$ are of the form σ_{i_k,j_k}, so that

$$\beta = \prod_{k=1}^{\mu} \sigma_{i_k,j_k}\,, \tag{2.5}$$

then they can be regarded as embedded bands. Links which arise this way, i.e., such with *positive band presentations*, are called *strongly quasipositive links*.

Bennequin's inequality [Be, Theorem 3] states

$$-\chi(L) \geq w - n \tag{2.6}$$

for an n-strand braid representative of L of writhe w. If there is a braid representative β of L making (2.6) an equality, we call both L and β *Bennequin-sharp*. This inequality was later extended to

$$-\chi(L) \geq -\chi_4(L) \geq w - n \tag{2.7}$$

(see e.g. [IS, St2]). In an analogous way we defined that L and β are *slice-Bennequin-sharp*.

It implies that a *strongly quasipositive surface*, i.e., obtained from a positive band presentation, is minimal genus. Namely, a positive band presentation of w bands on n braid strands gives a braid of writhe w. Thus the surface S constructed from the band presentation yields, with (2.7),

$$-\chi(L) \leq -\chi(S) = w - n \leq -\chi_4(L) \leq -\chi(L)\,.$$

This also shows that a strongly quasipositive link L is always Bennequin-sharp, and

$$\chi_4(L) = \chi(L)\,. \tag{2.8}$$

The *Bennequin sharpness conjecture* (see [FLL, St2]) asserts

$$L \text{ is Bennequin-sharp} \iff L \text{ is strongly quasipositive}\,. \tag{2.9}$$

For some related results, see [JLS].

The second author's effort to determine the quasipositivity of the (prime) 13 crossing knots [St4] also provides some evidence for a "4-ball" version of the Bennequin sharpness conjecture (2.9):

$$L \text{ is slice-Bennequin-sharp} \iff L \text{ is quasipositive}\,. \tag{2.10}$$

In practical terms, every proof of non-quasipositivity of a knot passes via showing that it is not slice-Bennequin-sharp. Note again that $\Longleftarrow$ follows immediately from (2.7), and it is $\Longrightarrow$ which is open.

Definition 2.1

- Let $b(K)$ be the braid index of K, the minimal number of strings of a braid representative of K.
- Let $b_b(K)$ be the Bennequin braid index of K, the minimal number of strings to span a Bennequin surface of K.
- When K is strongly quasipositive, let $b_{sqp}(K)$ be the minimal number of strings to span a strongly quasipositive surface of K (only positive bands).

- Further, for a Seifert surface S, let $b(S)$ be the minimal string number on which S is spanned as a braided surface.
- If S is a strongly quasipositive surface, let $b_{sqp}(S)$ be the minimal string number on which S is spanned as such (i.e., arises from a positive band presentation).

We have then (with the right inequality only valid for strongly quasipositive K)

$$b(K) \le b_b(K) \le b_{sqp}(K)\,, \tag{2.11}$$

and by definition, with S being a Seifert surface of K,

$$b_b(K) = \min\{\, b(S)\,:\, \chi(S) = \chi(K)\,\}\,, \qquad b_{sqp}(K) = \min\{\, b_{sqp}(S)\,:\, S \text{ strongly quasipositive}\,\}\,. \tag{2.12}$$

We will further discuss these relations in §5 and [JLS]. We also feature the following result. It confirms an expectation originally formulated for $n = b(L)$ by Jones [J, end of §8] (later also referred to as the "weak" form) and subsequently extended by Kawamuro.

Theorem 2.2 (proof of the Jones-Kawamuro conjecture [DP, LaM]) For every link L, there is a number $w_{min}(L)$, so that every braid representative β of L on n strands of writhe w satisfies

$$|w - w_{min}(L)| \le n - b(L)\,. \tag{2.13}$$

Generally speaking, we will use this theorem to advance theoretical applications in our work, but for practical ones, another tool will be crucial, the HOMFLY-PT polynomial. As announced, its treatise will be moved out to [JLS].

2.4 Slice-torus invariants

We briefly recall Livingston's [Lv] formalism of "slice-torus invariant". An integer-valued invariant v on knots (i.e., not necessarily defined for multi-component links) is a *slice-torus invariant* if

- $v(K) = -v(!K)$, and $v(-K) = v(K)$, where $-K$ is K with the reverse orientation
- additivity under connected sum: $v(K_1 \# K_2) = v(K_1) + v(K_2)$
- crossing switch rule: $v(K_+) - v(K_-) \in \{0, 1\}$
- $v(K) \le g_4(K)$ (or equivalently $2v(K) \le 1 - \chi_4(K)$), and
- v satisfies Bennequin's inequality:
$$2v(K) \ge w - n + 1$$
for a braid representative of K on n strings and writhe w.

These properties are not minimal (i.e., some follow from special cases or combinations of others), but they are what we will use in §6. (We emphasize that it is not assumed v to be defined on multi-component links $\kappa(K) > 1$ in any way.)

There are two known instances of such an invariant, the Ozsváth-Szabó τ invariant, and (half of) Rasmussen's invariant s. Thus the concept was introduced mainly to give these two a uniform treatment. (The halved *signature* $\sigma/2$ satisfies the first four of the above five properties, but not the last.)

From the superposition of

$$2g(K) = 1 - \chi(K) \ge 1 - \chi_4(K) \ge 2v(K) \ge w - n + 1$$

with (2.6), it is straightforward that

$$\text{if } v(K) < g(K), \text{ then } K \text{ is not Bennequin-sharp}. \tag{2.14}$$

In relation (see the remark below (2.10)), it follows that when a knot K is quasipositive, then

$$v(K) = g_4(K)\,, \tag{2.15}$$

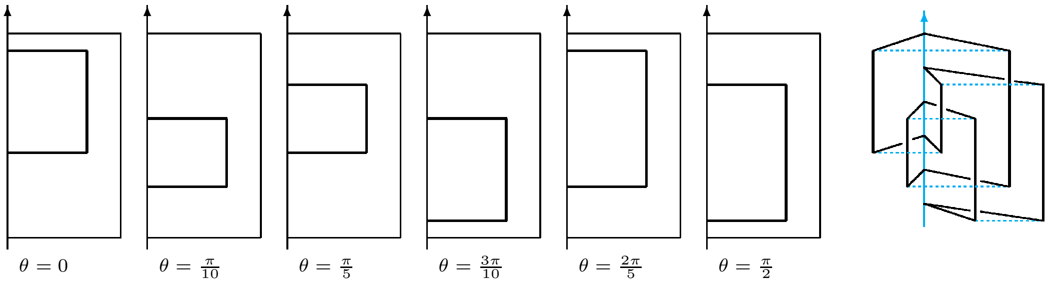
and

$$\text{when } K \text{ is strongly so, then } v(K) = g_4(K) = g(K). \tag{2.16}$$

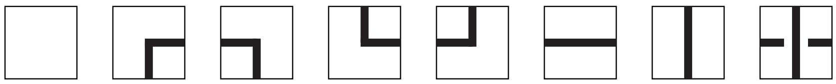
We will refer to these standard facts a few times below.

2.5 Grid diagrams and arc index

An *arc presentation* of a knot or a link L is an ambient isotopic image of L contained in the union of finitely many half planes, called *pages*, with a common boundary line in such a way that each half plane contains a properly embedded single arc.

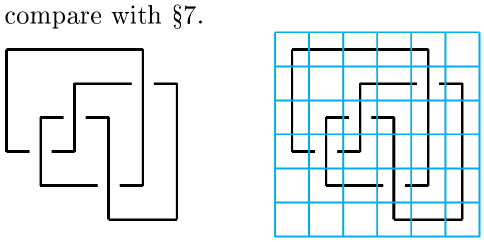


A *grid diagram* (or, for simplicity simply called *grid* often below) is a knot or link diagram which is composed of finitely many horizontal edges and the same number of vertical edges such that vertical edges always cross over horizontal edges. We assume that horizontal/vertical positions of vertical/horizontal edges are pairwise distinct. In particular, away from crossings edges only meet at corners, and vertices are pairwise distinct. Up to the adjustment of heights of horizontal and widths of vertical edges, a grid diagram is thus what can be composed in the plane by the tiles

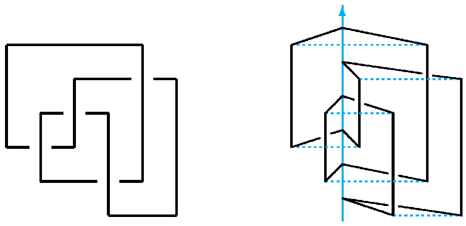


(2.17)

See, for example, [AL], and also compare with §7.



It is not hard to see that every knot admits a grid diagram (compare with the proof of Lemma 4.18). The figure below explains that every knot admits an arc presentation.



We set the *size* $\mu(D)$ of a grid diagram to be the number of vertical *or* (equivalently) horizontal segments (but *not both* together). A grid (diagram) of size μ will also be shortly called a μ-*grid*.

In general, we will afford the sloppiness of abolishing the distinction between an ordinary and a grid diagram, whenever the grid structure is unnecessary. Thus, for instance, $c(D)$ can mean the crossing number of both an ordinary and grid diagram, whereas $\mu(D)$ would imperatively assume that D is given a grid shape.

Let $a(L)$ be the *arc index* of L, the minimal $\mu(D)$ over all grid diagrams D of L. It is the minimal number of pages among all arc presentations of a link L [J+, J+2].

We note that the following was proved by Cromwell [Cr]. For two links L_1, L_2,

$$a(L_1\#L_2) = a(L_1) + a(L_2) - 2\,. \tag{2.18}$$

For knots L_i, it also follows from a relationship

$$a(K) = \lambda(K) + \lambda(!K)\,, \tag{2.19}$$

derived by Dynnikov-Prasolov [DP], concerning the Thurston–Bennequin invariant (see §4 for notation, Theorem 5.14 and Corollary 4.11), and the additivity of the invariant [EH, To].

2.6 Knot tables

For notation from knot tables, we follow Rolfsen's [Ro, Appendix] numbering up to 10 crossings, except for the removal of the Perko duplication.

For 11 to 16 crossings we use the tables of [HT] (which for 11 to 13 crossing knots are now also on KnotInfo [LvM]), while appending non-alternating knots after alternating ones of the same crossing number. Thus, for instance, $11a[k] = 11_{[k]}$ for $1 \le k \le 367$, and $12n[k] = 12_{1288+[k]}$ for $1 \le k \le 888$.

If it is relevant, mirror images will be distinguished on a case-by-case basis. Specifically, for the $(2,n)$-torus knots, we will say that the knot is *positively/negatively mirrored*. The convention for 10_{132} is fixed in Example 4.10. (This knot will archetype certain phenomena that become increasingly relevant in [JLS].)

3 Upper bound of Crossing number

A *planar grid polygon* can be defined as the planar projection of a link grid diagram. Similarly we specify its size by the number of horizontal/vertical edges. It is obvious that a grid polygon determines uniquely (when crossing convention is fixed) a grid diagram, and vice versa. A grid polygon can have multiple *components* (i.e., PL-immersed circles). Such objects are of interest in discrete geometry; see for example [BOS, SG, KLA]. They are named *rectilinear* polygons, but for us it is (very) relevant that self-crossings are allowed (i.e., the polygons are *not simple*).

Lemma 3.1 Every, possibly multiple-component, planar grid polygon of size m has the following upper bounds on the number of intersections.

$$\begin{cases} (m^2-2m)/2 & m \text{ is even} \\ (m-1)^2/2-1 & m \text{ is odd} \end{cases} \tag{3.1}$$

These bounds are sharp.

Proof. What could be a conundrum becomes self-evident after introducing the right way of counting crossings. We will group crossings w.r.t. their horizontal segment l. We consider the horizontal segments from the middle high segment upward. Whenever l is such a segment and l_h is a horizontal segment above

l, we define the *weight* $w_l(l_h) \in \{0,1,2\}$ to be the number of neighboring vertical edges of l_h intersecting l. Then

$$\#\{\,\text{intersections of } l\,\} = \sum_{l_h \text{ above } l} w_l(l_h)\,. \tag{3.2}$$

This counting works because for each vertical edge l_v intersecting l, exactly one of its two neighboring horizontal edges l_h is above l. Thus

$$\#\{\,\text{intersections of } l\,\} \le 2\#\{\,l_h\,:\,l_h \text{ above } l\,\}\,. \tag{3.3}$$

Now this sum will account to

$$\#\{\,\text{intersections of upper horizontal edges}\,\} \le 2\sum_{k=0}^{(m-1)/2} k \tag{3.4}$$

for the upper $(m+1)/2$ edges l when m odd. The lower $(m-1)/2$ edges l can be handled by choosing l_h to be below l, giving a similar sum

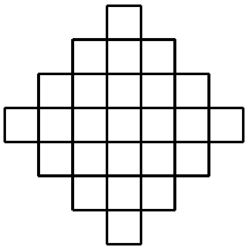
$$\#\{\,\text{intersections of lower horizontal edges}\,\} \le 2\sum_{k=0}^{(m-3)/2} k\,. \tag{3.5}$$

For m even one has $4\sum_{k=0}^{m/2-1} k$ instead of (3.4)+(3.5). Direct calculation gives

$$\#\{\,\text{intersections of polygon}\,\} \le \begin{cases} (m^2-2m)/2 & m \text{ is even} \\ (m-1)^2/2 & m \text{ is odd} \end{cases} \tag{3.6}$$

It remains to argue why for m odd, $(m-1)^2/2$ intersections are not possible. This would mean that the middle horizontal edge e satisfies $w_e(e') = 2$ for every other horizontal edge e'. But there are at least two edges e' for which this is not possible, namely those connected to the two vertical edges adjacent to e. This completes the proof of (3.1). For the sharpness of the bounds, see Example 3.2 and Proposition 3.3. □

Example 3.2 If we allow multiple components of the polygon, and consider m even, then the bound in (3.6) is sharp



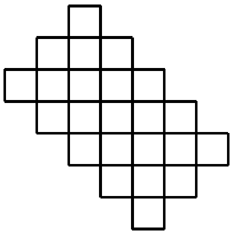
When we restrict to 1-component polygons, we know the following.

Proposition 3.3 For every $m > 2$, there exists a 1-component planar grid polygon Π_m of size m with

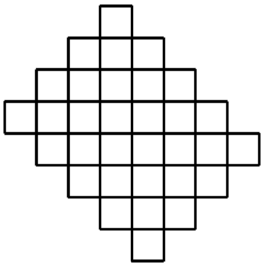
$$(m-1)^2/2 - \begin{cases} 1 & m \text{ is odd} \\ 5/2 & m \equiv 0 \bmod 4 \\ 7/2 & m \equiv 2 \bmod 4 \end{cases} \tag{3.7}$$

crossings.

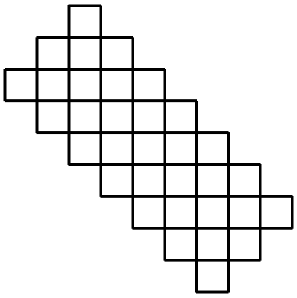
Proof. Consider the *Lissajous*[1] *polygon* $\Lambda(m_1, m_2)$.



$\Lambda(3,5)$

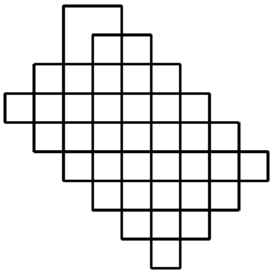


$\Lambda(4,5)$

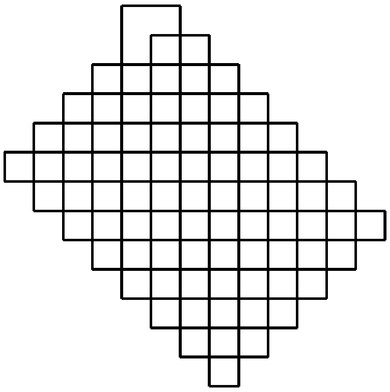


$\Lambda(3,7)$

This gives a grid polygon of size $m = m_1 + m_2$. When m_1 and m_2 are even and $m_1 - m_2 = 2$, we also need the *modified Lissajous polygon* $\Lambda'(m_1, m_2)$.



$\Lambda'(4,6)$



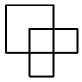
$\Lambda'(6,8)$

Then choose

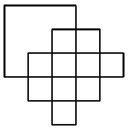
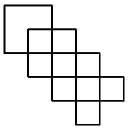
$$\Pi_m = \begin{cases} \Lambda(\frac{m-1}{2}, \frac{m+1}{2}) & m \text{ is odd} \\ \Lambda(\frac{m}{2} - 1, \frac{m}{2} + 1) & m \equiv 0 \bmod 4 \\ \Lambda'(\frac{m}{2} - 1, \frac{m}{2} + 1) & m \equiv 2 \bmod 4 \end{cases} . \tag{3.8}$$

The crossing number of these polygons (3.7) follows by explicit calculation. They are 1-component by direct inspection. (In general, $\Lambda(m_1, m_2)$ appears to be 1-component when m_1 and m_2 are relatively prime.) $\square$

These examples leave not much room for improvement. For odd m, (3.1) settles their maximality. When m is even, by modifying the argument at the end of the proof of Lemma 3.1 to the middle two horizontal edges, it is also easy to conclude that (3.1) cannot be made sharp by a 1-component polygon. Thus the examples of Proposition 3.3 can be improved by at most 1 crossing for $m \equiv 0 \bmod 4$ and by at most 2 crossings for $m \equiv 2 \bmod 4$.

Computation 3.4 Still, verifying whether the (1-component) polygons Π_m have maximal crossings (for even m) is not entirely trivial. We wrote a computer program to test this, which in fact found the family Λ' in (3.8). For $m = 4, 6$ there are exceptional maximal crossing polygons $\Lambda'(2,2) =$ 

[1]We chose this name since they appear as rectifications of Lissajous curves, although this correspondence is not precise for every (m_1, m_2).

(as compared to $\Lambda(1,3)$) , and  , as compared to $\Lambda'(2,4) =$  . We know that the polygons (3.8) have maximal crossings for even m with $8 \le m \le 24$.

Certainly, we are interested more in grid diagrams of links, with crossing information, and in that case Lemma 3.1 easily modifies to show the following.

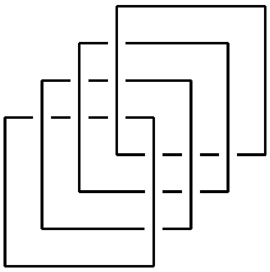
Lemma 3.5 Every grid link diagram D of size $\mu(D) = m$ has writhe $|w(D)| \le (m-1)^2/4$.

Proof. It is essentially the same proof as for Lemma 3.1, except noting that (3.2) modifies to

$$\sum (\text{ signs of intersections of } l\,) \le \sum_{l_h \text{ above } l,\ w_l(l_h)=1} w_l(l_h)\,, \tag{3.9}$$

because signs of crossings on l coming from l_h with $w_l(l_h) = 2$ cancel out. Then the estimates (3.3) and (3.6) exactly halve. □

Example 3.6 If we allow multiple components of the link, and consider m even, then again the bound in Lemma 3.1 (in the form of halving (3.6)) is sharp, as shows the $(m/2, m/2)$-torus link:



At the cost of decreasing the number of crossings by $O(m)$, one may obtain a knot, like the $(m/2, m/2+1)$-torus knot, or adjust m to be odd.

This is an immediate consequence of Lemma 3.1.

Corollary 3.7 For every link L,

$$c(L) < (a(L)-1)^2/2\,, \tag{3.10}$$

where $c(L)$ is the crossing number and $a(L)$ is the arc index. □

In contrast to the well-known bounds in [BP, JP] (see the proof of Corollary 5.5), it is striking that an *upper* estimate of the crossing number of a link in terms of its arc index has apparently never been previously considered in the literature.

Remark 3.8 While some reductions of the bound (3.10) may be possible under link isotopy (which is reflected in Cromwell's moves [AHT, Cr]), we have failed to significantly improve upon this estimate. This (combinatorial) problem develops serious enough to merit a separate account if later progress is made. We note, though, that in Example 3.6, the featured torus links appear in minimal crossing number diagrams, so that the right of (3.10) cannot be reduced by more than a factor of 2 (asymptotically in $a(L)$).

4 Thurston-Bennequin invariant

4.1 Weight model for the Thurston-Bennequin invariant

The main topic of this work starts from the observation that a braided surface of Euler characteristic 0, which is a K-knotted annulus, is essentially a grid diagram of the underlying companion knot K.

Definition 4.1 Let for a knot K and integer t,

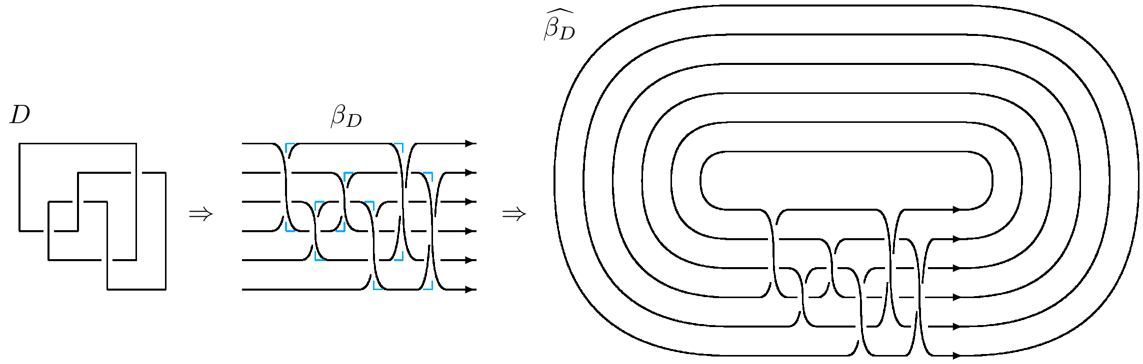
- $A(K,t)$ be the (link of the) t-framed K-knotted annulus,
- $W_+(K,t)$ and $W_-(K,t)$ the t-framed Whitehead doubles of K with positive and negative clasp, and
- $B(K,t)$ the t-framed Bing double of K.

We will usually abuse the distinction between the annulus and the link which is its boundary.

To disambiguate among different conventions for framing used elsewhere, we emphasize that t is here the linking number of the two components of $A(K,t)$. Thus, for example, $A(\bigcirc,1)$ is the positive (right-hand) Hopf link, and $A(\bigcirc,-1)$ the negative one. This definition of framing has the opposite sign to the one used by other authors (e.g., [DM]), where they take the writhe $w(D)=-t$ of a diagram D of K from which $A(K,t)$ is constructed as the blackboard-framed (reverse) 2-parallel.

Also, $W_+(\bigcirc,1)$ is the positive (right-hand) trefoil, and $W_+(\bigcirc,-1)=W_-(\bigcirc,1)$ the figure-8-knot. We can understand $W_+(K,t)$ resp. $W_-(K,t)$ as the result of plumbing a positive resp. negative Hopf band into $A(K,t)$ and taking the knot which is the boundary of the resulting Seifert surface. In a similar way, we can understand $B(K,t)$ as the 2-component link which is obtained by plumbing both a positive and a negative Hopf band into $A(K,t)$ and taking the boundary. Thus for instance $B(\bigcirc,0)$ is the 2-component unlink, and $B(\bigcirc,1)$ is the Whitehead link.

Let D be a grid diagram of a knot K. Replacing each vertical segment with a half twisted band as shown below, we get a braid in band presentation, denoted by β_D. Then the closure $\widehat{\beta_D}$ bounds a twisted annulus. Therefore $\widehat{\beta_D}=A(K,t)$ for some t.



(4.1)

This correspondence was previously noted by Rudolph (see [Ru3, Fig 1]) and later by Nutt (compare with [Nu, Theorem 3.1]), and will be referred to as *grid-band construction*.

Consider the situation that the band presentation is positive. Then obviously $A(K,t)$ for the resulting framing t is strongly quasipositive. A question is what is the framing t, which we will write as

$$t=\lambda(D)\,, \tag{4.2}$$

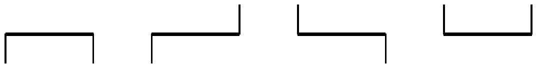
in dependence of the diagram D, and how to read $\lambda(D)$ off D. To explain the formula for $\lambda(D)$, given below as (4.5), we fix some notation.

Let the *weight* of a grid diagram D be

$$Z(D) = \frac{1}{2} \sum_{e \text{ edge of } D} \operatorname{sgn}(e)\,, \tag{4.3}$$

where the signs of the edges are determined as follows:

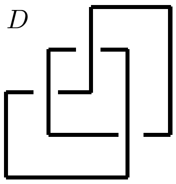
$$\operatorname{sgn}(e) = \begin{cases} 1 & e \text{ is vertical} \\ \pm 1,\ 0 & e \text{ is horizontal and one of the following forms} \end{cases}$$



$0 \qquad +1 \qquad -1 \qquad 0$ (4.4)

Example 4.2

D



$$Z(D) = \frac{1}{2}\left(\underbrace{(1+\cdots+1)}_{\text{vertical, L to R}} + \underbrace{(0+0+1+0+0)}_{\text{horizontal, B to T}}\right) = 3$$

Remark 4.3 This weight formula (4.3) can be generalized to non-positive band presentations by letting each vertical edge have the sign of the corresponding band. But we will treat this more general case only occasionally here.

We then can identify the framing t in (4.2).

Lemma 4.4 With $w(D)$ being the writhe, we have

$$\lambda(D) = Z(D) - w(D)\,. \tag{4.5}$$

Proof. One can see that when the Seifert circles of the closed braid diagram $\hat{\beta}_D$ of $A(K,t)$ are made small, one obtains a diagram of $A(K,t)$, where the band obtained by thickening D is twisted. By a straightforward combinatorial observation, the twisting of the band is given by $Z(D)$. The untwisted band built from D carries the framing $-w(D)$ itself. This gives (4.5). $\square$

Remark 4.5 One has a certain freedom to vary the direction from which to read the bands of the braid representative β_D of $A(K,t)$ off the grid diagram D of K. We explain our convention here in an example to make clear how band presentations are used for specific knots below. While horizontal and vertical edges are easily interchangeable in grids, disks and bands are far less so in braid band presentations. The change of direction will give different t, but will change K only up to mirroring.

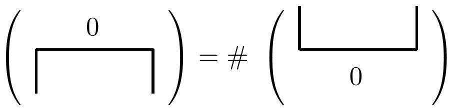
The default direction of reading will be from the left. Reading the grid diagram D in (4.1) from the left gives the word $[14][35][24][36][15][26]$, with $[ij] = \sigma_{i,j}$ of (2.2). Reading D from the right is meant to give the reverse order of (band) letters. This is the result of reading from the left a grid diagram D', which is obtained from the mirror image $!D$ after a flip (π rotation along the vertical axis). Reading D from the bottom gives the word $[46][25][13][24][36][15]$, which arises when reading from the left $!D$ after a rotation by $-\pi/2$. Reading from the top again reverses these letters and results in reading from the left D after the combination of a flip (along the horizontal axis) and $-\pi/2$ rotation. Note that thus we consider braid strands numbered right to left when reading a grid diagram from the right and from the top (while left to right otherwise).

Definition 4.6 Also, we can use the band presentation of β_D to specify the grid diagram D itself. The mirroring of D is fixed by default by saying that β_D should be obtained when reading D from the left. This means that we can write the grid diagram D in (4.1), even omitting brackets, as

```
14  35  24  36  15  26.
```

Since we deal with grids of size 10 or more, let us also already fix here that we use initial capital Latin letters `A,B,C,...` to denote two-digit integers $10, 11, 12, \ldots$, so that for example, `4C` $= [4, 12] = \sigma_{4,12}$.

Let $br(D)$ be the *vertical bridge number* of D, which is the number of sign-0 horizontal edges of D of one of either types in (4.4)

$br(D) := \#$ 

Lemma 4.7 We have

$$br(D) \le Z(D) \le \mu(D) - br(D)\,, \tag{4.6}$$

and thus

$$br(D) - (\mu(D) - 1)^2/4 \le \lambda(D) \le (\mu(D) + 1)^2/4 - br(D)\,. \tag{4.7}$$

Proof. The left inequality in (4.6) holds because each piece of the knot between two vertical extrema contributes at least 1 to the weight sum, and we have $2br(D)$ such pieces. The right inequality holds because there are $2br(D)$ edges in D with sign 0. By using Lemma 3.5,

$$-(\mu(D) - 1)^2/4 \le w(D) \le (\mu(D) - 1)^2/4\,. \tag{4.8}$$

Then, with (4.6), (4.5) and (4.8), we obtain (4.7). $\square$

Let $a(K)$ be the arc index of K, the minimal $\mu(D)$ over all grid diagrams D of K. Take a minimal grid diagram $\mu(D) = a(K)$. Then, with (4.6), (4.5) and (4.8), we have

$$br(D) - (a(K) - 1)^2/4 \le \lambda(D) \le (a(K) + 1)^2/4 - br(D)\,.$$

Thus we have, using the bridge number $br(K)$ of K from §2.2:

Theorem 4.8 There exists a number $\lambda_{min}(K)$ with

$$br(K) - (a(K) - 1)^2/4 \le \lambda_{min}(K) \le (a(K) + 1)^2/4 - br(K)\,, \tag{4.9}$$

such that for all $t \ge \lambda_{min}(K)$, we have that $A(K,t)$ is strongly quasipositive on $b_{sqp}(A(K,t)) \le a(K) + t - \lambda_{min}(K)$ strands.

We will use $\lambda_{min}(K)$ often in the following. Two caveats are in order regarding this notation. The 'min' refers to the minimum with respect to number of strings of the surface $A(K,t)$ (or horizontal segments in the grid diagram of K), *not* the framing t itself. And, it is *not* assumed that λ_{min} is unique. At least for the unknot K,

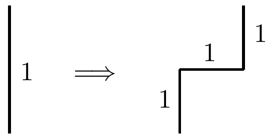
$$\text{both } b(A(\bigcirc, 0)) = b(A(\bigcirc, 1)) = 2, \text{ thus } \lambda_{min}(\bigcirc) = 0, 1. \tag{4.10}$$

This special behavior of unknot will require repeated attention. For a non-trivial knot K, the uniqueness and minimality of $\lambda_{min}(K)$ was settled, as will be discussed below; see Theorem 5.14. But we do not wish to exclude $K = \bigcirc$ consistently. We prefer to maintain the symbol $\lambda_{min}(K)$, stipulating that formulas involving $\lambda_{min}(K)$ are meant to hold whatever of either values (4.10) is chosen for $K = \bigcirc$. For $K \ne \bigcirc$, the reader may assume that

$$\lambda_{min}(K) = \lambda(K)\,, \tag{4.11}$$

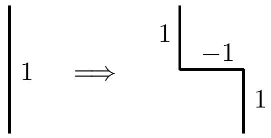
though we will not use this before stating Theorem 5.14.

Proof of Theorem 4.8. When μ is augmented by 1, we can always augment by 1,



(4.12)

resp. preserve



(4.13)

any given framing $\lambda(D)$ by the above two moves. (Apply an adjusting PL-map on the half-planes above and below the newly formed horizontal edge.) We call these moves in the following positive and negative *stabilization*, resp. Thus, $\lambda(D)$ augments by 1 under positive stabilization, and negative stabilization does not change $\lambda(D)$. (Neither stabilization changes $w(D)$. Note that the diagram D_1 of $A(K,t)$ obtained from D always has $s(D_1) = \mu(D)$ Seifert circles and writhe $w(D_1) = \mu(D)$.) The claim follows for $a(K) + t - \lambda_{min}(K)$ strands from positive stabilization, and for larger strand number by (further) negative one. $\square$

The *Thurston-Bennequin invariant* $TB(D)$ of a grid diagram D can be defined as is being identified in the following theorem.

Theorem 4.9 For any grid diagram D, the quantity $Z(D)$ counts the NW- or SE-corners of D.

$$Z(D) = \#\left(\ulcorner \quad \text{NW-corners} \right) = \#\left(\lrcorner \quad \text{SE-corners} \right) \tag{4.14}$$

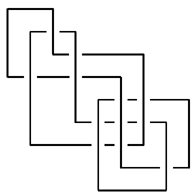
Thus (1.1) holds.

Proof. The argument for (4.14) is a combinatorial observation.

Obviously (4.14) is a claim about planar rectilinear (1-component) polygons (§3). Thus crossings of the grid diagram can be ignored. Every 1-component planar rectilinear polygon can be reduced to a planar rectangle by polygonal removal of (a) rectilinear kinks (Reidemeister I move) and (b) horizontal segments as in a (positive or negative) destabilization (4.12) and (4.13), and its $\pm\pi/2$-rotated versions for vertical segments. (Note that these moves can be applied now, up to some planar adjustment, even if the removed segments have intersections on them.) It is easy to observe that (4.14) is invariant under these simplifications, and holds for the rectangle.

Once we know (4.14), we obtain (1.1) straightforwardly from the characterization of $TB(D)$ given in [LvN] or [Ng] (or see [Ru3, p. 159]). $\square$

Example 4.10 The [J+] diagram D' of 10_{132},



10_{132}

read from the right (see Remark 4.5), gives the 9-strand band presentation

$$\beta_D = [25][14][37][26][15][48][79][38][69]\,, \tag{4.15}$$

where $D = \mathrm{flip}\,(!D')$. We have $\mu(D) = 9$, $Z(D) = 3$, $w(D) = 2$, $br(D) = 3$ and $\lambda(D) = 1$. Thus (4.15) gives a (positive) band presentation of $A(10_{132}, 1)$. The mirroring of 10_{132}, determined by D, is so that it *has the HOMFLY-PT polynomial of the positively mirrored* 5_1. We fix this mirroring in the sequel, also for later reference in [JLS]. Note that it is thus *opposite* to Rolfsen's [Ro, Appendix] mirroring.

We also remark the following straightforward consequence of Theorem 4.9.

Corollary 4.11 When the grid diagram $!D$ is obtained from D by switching all crossings, and a $-\pi/2$ rotation, then $\lambda(D) + \lambda(!D) = \mu(D)$.

Proof. The writhe terms of D and $!D$ in (4.5) cancel out. Thus $\lambda(D)+\lambda(!D) = Z(D)+Z(!D)$. By taking the average of the two corner counts in (4.14) for D and its $-\pi/2$ rotation, we see that $Z(D) + Z(!D)$ is half the number of all corners of D, which is $\mu(D)$. □

4.2 Application to strong quasipositivity

Let $TB(K)$ be the *maximal Thurston-Bennequin invariant* of K, an invariant often considered in contact geometry [FT, LvN, Ng, Ma, Ru3]:

$$TB(K) \;:=\; \max\{\, TB(D) \,:\, D \text{ is a diagram of } K\,\}\,.$$

We also specify a region which will play an important role throughout the rest of the paper (and beyond).

Definition 4.12 We define the *framing diagram* $\Phi(K)$ of K as a subset of $\mathbb{R}^2$ by

$$\Phi(K) \;:=\; \{\,(\mu,t) \,:\, A(K,t) \text{ has a strongly quasipositive band representation on } \mu \text{ strands}\,\}\,.$$

The following result of Rudolph [Ru3, Proposition 1] then follows directly from Theorem 4.9. (Note our different sign convention for t.)

Corollary 4.13 When K is not the unknot, then

$$\lambda(K) := \min\{\, t : \, A(K,t) \text{ is strongly quasipositive }\} = -TB(K)\,, \tag{4.16}$$

and more precisely,

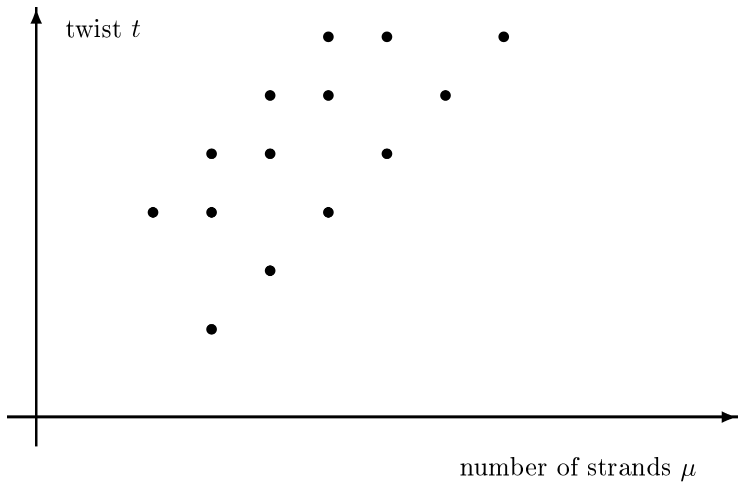
$$A(K,t) \text{ is strongly quasipositive} \iff t \ge -TB(K)\,. \tag{4.17}$$

Proof. When $A(K,t)$ has a positive band presentation, then every disk is connected by at least two bands. Disks connected by one band can be removed, and such connected by no band do not exist unless $K = \bigcirc$ (and $t = 0$), which was excluded. Since $\chi(A(K,t)) = 0$, it follows that every disk is connected by exactly two bands, which means that the band presentation of $A(K,t)$ gives rise to a grid diagram of K. It is also well known that every integer $t \ge -TB(K)$ can be realized as Thurston-Bennequin invariant of some grid diagram of K. (We mentioned this above; see (4.12).) Note in passing that undoing the removal of disks connected by one band is, up to conjugacy, *positive braid stabilization*. This also shows that

$$(\mu,t) \in \Phi(K) \Longrightarrow (\mu+1,t) \in \Phi(K)\,, \tag{4.18}$$

which equals the effect of the *negative grid* stabilization (4.13). □

The following diagram illustrates the effect of the *positive* grid stabilization within $\Phi(K)$:



For the unknot K, we have

$$-TB(\bigcirc) = 1 \text{ but } \lambda(\bigcirc) = 0\,. \tag{4.19}$$

The problem with (4.16) there is that $A(K,0)$ has the empty positive band presentation (on two strands), but we do not consider this band presentation corresponding to a grid diagram. For this reason, the unknot will repeatedly require special attention below. Despite the identification (4.16), $\lambda(K)$ will occur so often, that it is better to maintain the notation and avoid writing the minus sign most of the time, even when we exclude $K = \bigcirc$.

Remark 4.14 It is possible to derive similar properties for links K. Then a framing t is needed for each component, and the relationship in Corollary 4.13 becomes slightly more involved, as become the framing diagram of Definition 4.12 and its properties. We do not wish to deal extensively with (banded) links here. In the case of torus links, the second author has begun a separate collaboration project [MS+]. However, in situations where the surface structure is forgotten, more self-contained extensions to links K do emerge, as for Corollaries 5.4, 5.5, and 5.6.

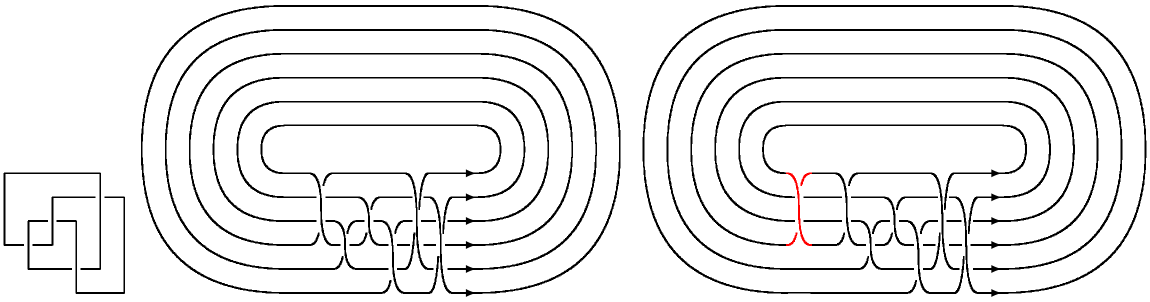
In the following application we assume that $K \neq \bigcirc$. For $K = \bigcirc$, all the links in Definition 4.1 are (alternating) 2-bridge links, and such can be handled *ad hoc* for strong quasipositivity (see e.g. [Ba]).

Corollary 4.15 Let K be a non-trivial knot. Then
(a) $W_+(K,t)$ is strongly quasipositive if and only if $t \geq -TB(K)$, and
(b) $W_-(K,t)$ and $B(K,t)$ are never strongly quasipositive.

Proof. The minimal genus surface of $W_\pm(K,t)$ is unique. This is proved by Whitten [Wh], but follows also from a result of Kobayashi [Ko]: the plumbing $S_1 * S_2$ is a unique minimal genus surface if and only if one of $S_{1,2}$ is a unique minimal genus surface and the other one is a fiber surface. The minimal genus surface of $W_\pm(K,t)$ is a Hopf band plumbed into the annulus $A(K,t)$ (which is obviously the unique minimal genus surface of $A(K,t)$; compare below Definition 4.1). Kobayashi's version also shows that plumbing two Hopf bands into $A(K,t)$ gives a unique minimal genus surface for $B(K,t)$. It follows from Rudolph's results on Murasugi sum [Ru2] that $W_-(K,t)$ and $B(K,t)$ are never strongly quasipositive: their unique minimal genus surface Murasugi desums into pieces, not all of which are strongly quasipositive. Also, $W_+(K,t)$ is strongly quasipositive if and only if $A(K,t)$ is. (For W_- see also the proof of Corollary 6.4.) $\square$

Since we will need this repeatedly later, let us already here notice that the Hopf plumbing $W_+(K,t) = A(K,t) * H$ can be realized by doubling a(ny) positive band in a band presentation of $A(K,t)$.

Example 4.16



4_1 $A(4_1,3)$ $W_+(4_1,3)$

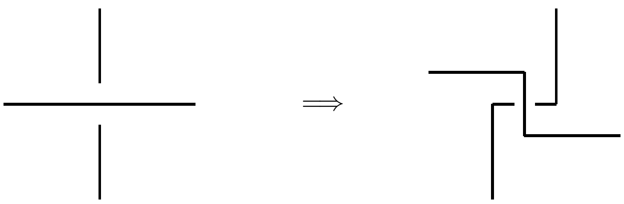
A similar remark applies to $W_-(K,t)$ whenever a band presentation of $A(K,t)$ has a negative band. However, it is important to note that this is not the only way to generate positive band presentations of Whitehead doubles. (A different example for a trefoil Whitehead double is given in [Be, fig p. 121 bottom].) We will discuss Whitehead doubles further in §6.

Here, we give a simple application of the weight model, in estimating the Thurston-Bennequin invariant.

Definition 4.17 Define $pbr(D)$, the *plane-bridge number* of D as the minimal number of Morse maxima (or minima, i.e., half of the minimal number of Morse extrema) over all smooth diffeomorphic images of D in S^2. Obviously $br(K) \leq pbr(D)$.

Lemma 4.18 For any diagram D of K, we have $\lambda(K) \leq 2c_-(D) + pbr(D)$.

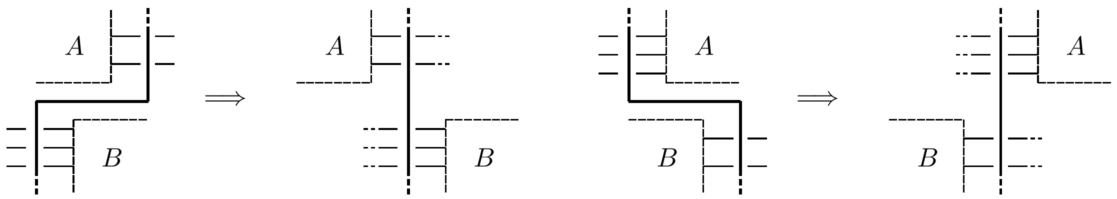
Proof. First, we can make D into a grid diagram by straightening out edges, and replacing wrong crossings, i.e., those with the horizontal strand on top, as follows.



(4.20)

(Again, as for the stabilization moves that occurred in the proof of Theorem 4.8, some small PL adjustment is needed.) This does not change the number of bridges.

Next, the horizontal adjustment technique (4.21) can be used to delete a horizontal edge e of label ± 1 without crossing on it, again by applying a suitable PL-map on the half-planes above and below e. This is the reverse of the stabilization moves, but we may need in advance to displace possible vertical edges above or below e. (If necessary, extend the box A resp. B drawn in the following figures until above resp. below the entire grid diagram, to ensure that all edges enter the box horizontally.)



(4.21)

The inequality we wish to prove about the diagram resulting after a move (4.21) is equivalent to the one about the original diagram. We may therefore assume henceforth that all ± 1 signed horizontal edges are intersected by a crossing. Thus

$$c(D) \geq \mu(D) - 2br(D)\,. \tag{4.22}$$

Also, we can see

$$Z(D) \leq \mu(D) - br(D)\,,$$

by using that in (4.3) there are $2br(D)$ edges of label 0.

Then

$$\begin{array}{rcl} \lambda(K) & \leq & Z(D) - w(D) \\ & \leq & \mu(D) - br(D) - w(D) \\ & \leq & c(D) + br(D) - w(D) \\ & = & 2c_-(D) + br(D)\,. \end{array}$$

The rest follows by minimization. $\square$

A counterpart to Lemma 4.18 will emerge later from the HOMFLY-PT polynomial, and is only announced here.

Lemma 4.19 ([JLS]) For any diagram D of K, we have $\lambda(K) > -2c_+(D) - pbr(D)$.

5 Braid indices

We discuss some relation regarding the braid indices in Definition 2.1. (Remember the comparison with [Ru3, Fig. 1] and [Nu, Section 3.3].) As noticed, Bennequin's inequality (2.6) implies that a strongly quasipositive surface is a Bennequin surface, thus for K strongly quasipositive, we have (2.11). We know that $b_b(K) > b(K)$ is possible [HS], but the examples K known are not strongly quasipositive. Rudolph conjectures that

$$b_{sqp}(K) = b(K) \tag{5.1}$$

when K is strongly quasipositive, and this is true, among other families, if K is a prime knot of up to 16 crossings (see [St2]). By the proof of the Jones-Kawamuro conjecture (Theorem 2.2), a Bennequin surface of a strongly quasipositive link K on $b(K)$ strands is always strongly quasipositive, so that

$$b_b(K) = b(K) \tag{5.2}$$

implies (5.1) for strongly quasipositive knots K. The problem (5.2) is extensively studied in [St2].

Since a band presentation of $A(K,t)$ always comes from a grid diagram of K, and with a confirmative notice about the unknot, we have:

Corollary 5.1

$$\min\{b_b(A(K,t))\,:\,t \in \mathbb{Z}\} = a(K)\,. \tag{5.3}$$

Moreover, there are at least $a(K)+1$ consecutive integers t which realize the minimum.

Proof. The case that K is the unknot can be handled directly: the minimizing t are $t=-1,0,1$.

When K is not the unknot, every maximal Euler characteristic (equal to 0) band presentation of $A(K,t)$ comes from a grid diagram of K. This shows

$$\min\{b_b(A(K,t))\,:\,t\in\mathbb{Z}\,\}\geq a(K)\,.$$

To see the reverse inequality, take a minimal grid diagram D of K. This gives a positive band presentation β_D of $A(K,t)$ for $t=\lambda_{min}(K)$. Now consecutively turn the $a(K)$ bands negative, which gives band presentations of Bennequin surfaces for $A(K,t)$ where $t=\lambda_{min}(K),\dots,\lambda_{min}(K)-a(K)$. □

Also, because choosing positive bands will give a band presentation of a strongly quasipositive annulus, we have with Corollary 4.13:

Corollary 5.2 $\min\{b_{sqp}(A(K,t))\,:\,t\geq\lambda(K)\,\}=a(K)\,.$ □

Forgetting the surface structure then yields an inequality of (ordinary) braid indices:

Corollary 5.3

$$\min\{b(A(K,t))\,:\,t\in\mathbb{Z}\,\}\leq\min\{b(A(K,t))\,:\,t\geq\lambda(K)\,\}\leq a(K) \tag{5.4}$$

Moreover, there are at least $a(K)+1$ consecutive integers t which realize the inequality $b(A(K,t))\leq a(K)$. □

The braid index of a link $A(K,t)$ is obviously not less than the sum of the braid indices of constituent components. Thus from Corollary 5.3, we also immediately have an inequality, which was noticed by Cromwell [Cr] (with the extension to links K as explained in Remark 4.14):

Corollary 5.4 (Cromwell) For every knot K, we have $2b(K)\leq a(K)$. □

We obtain then the (slight) refinement of Ohyama's inequality [Oh], as also explained in the introduction.

Corollary 5.5 For every knot K, we have $b(K)\leq c(K)/2+1$, and if K is non-alternating, then $b(K)\leq c(K)/2$.

Proof. It is known that $a(K)\leq c(K)+2$, as proved in [BP], and $a(K)\leq c(K)$ for K non-alternating [JP]. □

Since $b(K)\geq br(K)$, it further follows:

Corollary 5.6 For any knot K, we have $2(br(K)-1)\leq c(K)$. If K is non-alternating, then $2br(K)\leq c(K)$. □

In the obvious extension to links, connected sums of Hopf links show that the (first) inequality is sharp. But there is a more precise conjecture, apparently due to Fox [Fo], and later studied and extended by Murasugi [Mu]. For *knots* K, it states

$$3(br(K)-1)\leq c(K)\,.$$

These useful implications are worth noting, but we will see below that it is much more important to work with (5.4) rather than its simplified variant of Corollary 5.4.

We are next going discuss what (say, strongly quasipositive) framings λ are possible for given grid size μ, and in particular whether λ_{min}, the framing for a minimal (size $a(K)$) diagram (see Theorem 4.8) is unique. Since μ bounds the braid index of $A(K,t)$, and all have the same χ, Birman-Menasco [BM] imply that for given λ, only finitely many μ are possible. We will later prove in [JLS] a more precise statement (Finite-Cone-Theorem 5.11).

Question 5.7 (a) Is $b(A(K,t)) \geq a(K)$ for any t?
(b) At least, is $b(A(K,t)) \geq a(K)$ for any strongly quasipositive $A(K,t)$?

We reformulate part (a) here as a conjecture, with the insight gained from Corollary 5.3.

Conjecture 5.8

$$a(K) = \min_{t\in\mathbb{Z}} b(A(K,t)) \tag{5.5}$$

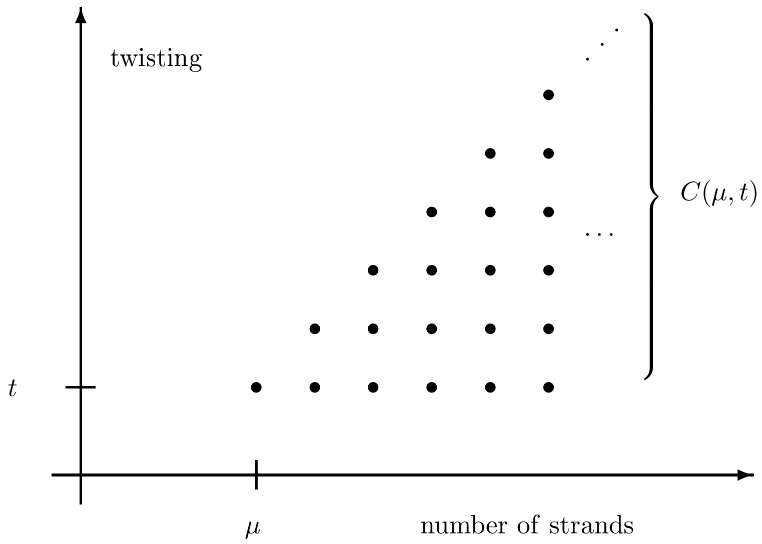
If part (b) fails, then it would give an example $A(K,t)$ answering negatively Rudolph's question (5.1).

To formalize this topic better, we introduce notation relating to the two grid stabilizations (4.12) and (4.13).

Definition 5.9 We define the *cone* $C(\mu,t) \subset \mathbb{Z}_+ \times \mathbb{Z}$ by

$$C(\mu,t) = \{\ (s,\lambda) :\ \ s \geq \mu,\ \ t \leq \lambda \leq t+s-\mu\ \}\,.$$

We say (μ,t) is the *tip* of the cone.



We can summarize some properties we have derived regarding the region $\Phi(K)$ of Definition 4.12.

Theorem 5.10 (a) The framing diagram $\Phi(K)$ of K (see Definition 4.12) is a union of cones.
(b) It contains at least one cone of the form $C(a(K),\lambda_{min}(K))$ and one of the form $C(\mu,\lambda(K))$.
(c) It contains no points with $t < \lambda(K)$ and $\mu < a(K)$.
(d) Every point $(\mu,t) \in \Phi(K)$ satisfies

$$br(K) - (\mu-1)^2/4 \leq t \leq (\mu+1)^2/4 - br(K)\,. \tag{5.6}$$

□

This estimate (5.6), that comes from (4.7), is rather crude, due to our insufficient control over the writhe. One problem with (4.8) is that, while it can be (at least asymptotically) sharp on either side, this unlikely happens (simultaneously) for diagrams D of the same link. Methods to address the writhe variation exist, based on Thistlethwaite's work on the Kauffman polynomial, but they will lead to no pleasant results here. A far more efficient technique will be introduced later in [JLS], which ultimately leads to much sharper bounds than (5.6), especially when K is fixed and μ is large. However, we emphasize that neither (5.6), nor the inequalities in Lemmas 4.18 and 4.19, follow from alternative estimates we obtain (or, to the best of our knowledge, other known results).

We announced that we will prove later in [JLS] the Finite-Cone-Theorem.

Theorem 5.11 (Finite-Cone-Theorem [JLS]) The framing diagram $\Phi(K)$ is a union of finitely many cones.

The following Jones-Kawamuro type of conjecture (compare with Theorem 2.2) is then suggestive.

Question 5.12 If K is non-trivial, is $\Phi(K)$ a single cone? (This cone would have to be then $C(a(K), \lambda_{min}(K))$ with $\lambda_{min}(K) = \lambda(K)$.)

Example 5.13 According to (4.10), we have

$$\Phi(\bigcirc) = C(2,0) \cup C(2,1)$$

being the union of two cones.

The special case for $\mu = a(K)$ in Question 5.12 (an analogue of the "weak" form of the Jones-Kawamuro conjecture) was already raised in [Ng] in the language of grid diagrams D and Thurston-Bennequin invariants $TB(D)$. It was answered in [DP, Corollary 3].

Theorem 5.14 (Dynnikov-Prasolov [DP]) The Thurston-Bennequin invariant of minimal grid diagrams of a given knot K is always equal to $TB(K)$.

We will return to this statement in [JLS]. Note that the unknot creates no exception here, when using TB instead of λ and avoiding the discrepancy (4.19).

6 Jump invariant

Turning to Whitehead doubles, Ozsváth-Szabó defined a number $j_\tau(K)$, the *jump invariant* of τ, with

$$\tau(W_+(K,t)) = \left\{ \begin{array}{ll} 1 & t \geq j_\tau(K) \\ 0 & t < j_\tau(K) \end{array} \right. . \tag{6.1}$$

The existence of such a number can be seen easily from Livingston's properties of slice-torus invariants (§2.4). We have $g(W_+(K,t)) = 1$, so for strongly quasipositive $T = W_+(K,t)$ we have $\tau(T) = 1$ (see (2.16)). Also $W_+(K,t) \to W_+(K,t-1)$ and $W_+(K,t) \to \bigcirc$ by a positive-to-negative crossing change, thus $\tau(T) \in \{0,1\}$. It is not immediately clear that $\tau \not\equiv 1$, i.e., $j_\tau(K) > -\infty$, but this is known, and we will also be able to derive it in Proposition 6.3.

It is important, for reasons (6.14) that will transpire below, that τ can be replaced by (half of) Rasmussen's invariant s, or any other (possible) slice-torus invariant v. In particular, for any such v we have the behavior of (6.1), leading to defining the *jump number* $j_v(K)$, as studied in [LvN]. In fact, note that one can define j_σ for the signature σ as well (after some modification $(\sigma+1)/2$ to fit values $0,1$), but for obvious reasons $j_\sigma(K) = 1$ regardless of K. Corollary 4.15 shows then that there are many Whitehead doubles T which are not strongly quasipositive despite $\sigma(T) = 2g(T) = 2$. We obtain then the following.

Corollary 6.1 For any slice-torus invariant v, we have

$$j_v(K) \leq \lambda(K)\,. \tag{6.2}$$

Proof. By Corollary 4.15, $W_+(K,t)$ is strongly quasipositive for $t \geq \lambda(K)$, thus from (2.16), we have $v(W_+(K,t)) = 1$ for $t \geq \lambda(K)$. □

Example 6.2 Equality does not always hold. An example for $v = \tau$ is $T = W_+(!3_1,3) = 14_{45575}$, which is a Whitehead double of the negative (left-hand) trefoil $!3_1$. There $\tau(T) = 1$, but $T = 14_{45575}$ is not strongly quasipositive. We have $\lambda(!3_1) = 6$ (see [LvM]).

Now we can also easily recover the Livingston-Naik result [LvN].

Proposition 6.3 For any slice-torus invariant v, we have

$$-\lambda(!K) < j_v(K) \leq \lambda(K)\,. \tag{6.3}$$

Proof. The right inequality in (6.3) was given in Corollary 6.1. To obtain the left inequality, we prove that

$$v(W_+(K,t)\#W_+(!K,-t)) \leq 1\,. \tag{6.4}$$

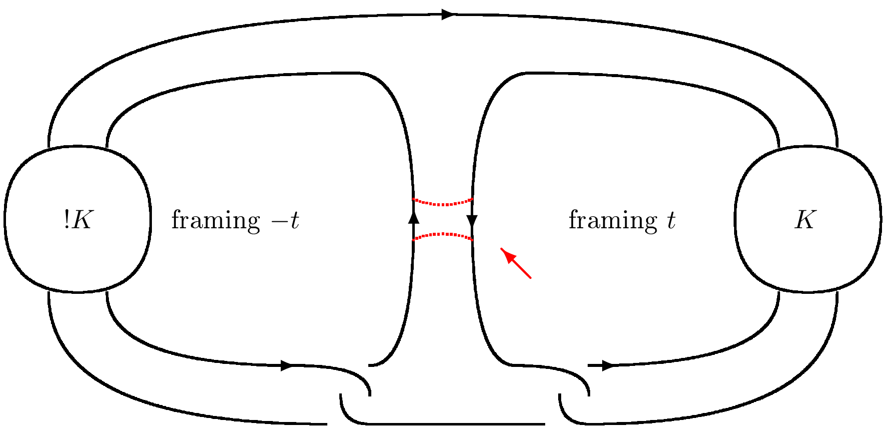
We remind that both connected sum factors have v-invariant 0 or 1.

Assume (6.4) is proved. Then since $v(W_+(!K,-t)) = 1$ for $-t \leq -\lambda(!K)$ for the same reasons as Corollary 6.1, we need from (6.4) that $v(W_+(K,t)) = 0$ for $t \leq -\lambda(!K)$, so we have the left inequality in (6.3).

To prove (6.4), assume by contradiction that l.h.s. is 2. Thus $\chi_4(W_+(K,t)\#W_+(!K,-t)) \leq -3$.

By connecting with a band as indicated in Figure 1, we obtain a 2-component link in Figure 2, with presumably

$$\chi_4[(6.7)] \leq -2\,. \tag{6.5}$$



(6.6)

Figure 1: Splice at the place indicated by the arrow, by adding a band

But the disk region of (6.7) represents an annulus of the slice knot $K\#!K$ with framing $t - t = 0$. However, pay attention that there is an orientation issue here. When K is non-invertible, then $K\#!K$ is slice only if $!K$ is oriented in a proper way. To resolve this issue, notice that the construction of $W_+(K,t)$ does not depend on the orientation of K, and moreover, $W_+(K,t)$ is easily seen to be invertible regardless of whether K is or not. This means one can suitably choose orientations of $W_+(K,t), W_+(!K,-t)$ when their connected sum in (6.4) is built. The v invariant obviously is not affected by this choice. Then by smoothing out any one of the four displayed crossings in (6.7), we obtain the unframed Whitehead double (6.8) $= W_+(K\#!K,0)$ of a slice knot, in Figure 3, which must be slice itself and thus have $\chi_4 = 1$. But from (6.5), we would need $\chi_4[(6.8)] \leq -1$, a contradiction. $\square$

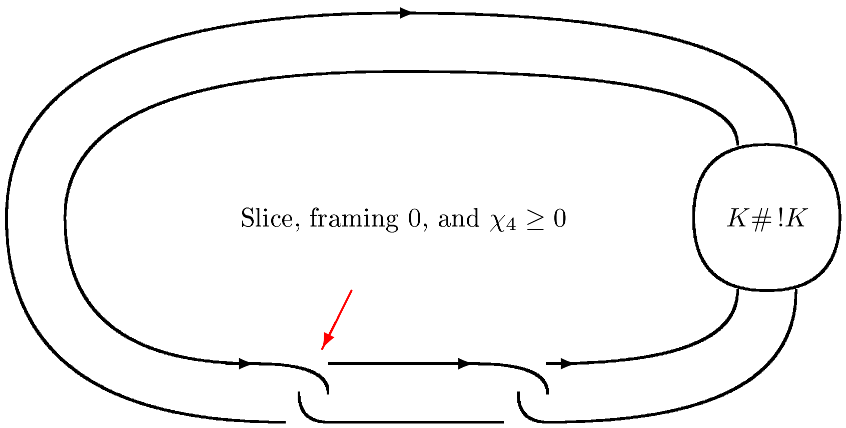
We have then the following contribution to the Bennequin sharpness conjecture (2.9).

Corollary 6.4 Assume there is a slice-torus invariant v so that (6.2) is sharp for K:

$$\lambda(K) = j_v(K)\,. \tag{6.9}$$

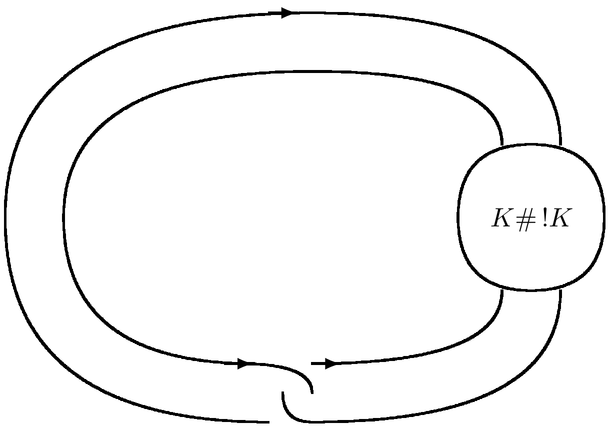
Then for every t,

$$W_\pm(K,t) \text{ is Bennequin-sharp} \iff W_\pm(K,t) \text{ is strongly quasipositive}\,. \tag{6.10}$$



(6.7)

Figure 2: One of the four crossings should be smoothed out, and then one nugatory crossing removed



(6.8)

Figure 3: Slice knot, $\chi_4 = 1$

Proof. If K is the unknot, then $W_\pm(K,t)$ are twist knots, so alternating, and for them (2.9) is resolved; see [FLL, St2]. (Or one can make an explicit check.) Thus we assume below that $K \neq \bigcirc$.

We first deal with W_+. If (6.9) holds, then because of Corollary 4.15,

$$W_+(K,t) \text{ is not strongly quasipositive } \iff v(W_+(K,t)) = 0\,. \tag{6.11}$$

Furthermore, $g(W_+(K,t)) = 1$, thus by (2.16),

$$v(W_+(K,t)) = 0 \Longrightarrow W_+(K,t) \text{ is not Bennequin-sharp}\,. \tag{6.12}$$

Combining (6.11) and (6.12) gives the '$\Longrightarrow$' direction in (6.10). The reverse direction,

$$\text{not Bennequin-sharp} \Longrightarrow \text{not strongly quasipositive,} \tag{6.13}$$

is among the standard causalities following from Bennequin's inequality (2.6) (see the remark above (2.8)).

For W_- notice that it unknots by a negative-to-positive crossing change, so that $v(W_-) \leq 0$, while $g(W_-) = 1$ (unless K is the unknot, and $t = 0$, a case that can be handled extra). Thus $W_-(K,t)$ cannot be Bennequin-sharp by (2.14). Then neither can it be strongly quasipositive by (6.13). Compare with Corollary 4.15(b), for which this reasoning thus gives an alternative proof.

This means that (6.10) holds for W_- as an equivalence of false assertions for whatever K (and t), regardless of the condition (6.9). □

Of course, when v is effectively computable, so is $j_v(K)$. But at least for $v = \tau$, there is a more closed expression. Hedden [He] has found that

$$j_\tau(K) = 1 - 2\tau(K)\,, \tag{6.14}$$

which further elucidates Example 6.2. But the picture for Rasmussen's invariant remains less clear.

We can fit (6.14) into the general relationship

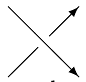
$$\lambda(K) = -TB(K) \geq 1 - 2\tau(K) = j_\tau(K) \geq \chi_4(K)\,. \tag{6.15}$$

For the leftmost inequality, which is due to Plamenevskaya, see the proof of Theorem 1.5 in [He2]. One can use (6.15) to easily obtain the property (6.3) for $v = \tau$, which motivated treating there a general v rather than only focussing on this special instance. The relationship (6.14) also identifies when (6.9) holds for $v = \tau$, namely which occurs when

$$\lambda(K) = 1 - 2\tau(K)\,. \tag{6.16}$$

This raises the question what knots satisfy (6.16). There is one noteworthy class.

Lemma 6.5 Every positive knot K satisfies (6.16).

Proof. A positive diagram D of K can be Morsified as done by Tanaka [Ta2]. One can view a positive diagram D, with $w(D) = c(D)$, as a front diagram; we put each positive crossing so that it is in the form  , and put each Seifert circle to form a front diagram, so that it contributes exactly one left and exactly one right cusp. Thus $w(D) = c(D)$ and in (4.5) (after a $-\pi/4$ rotation) $Z(D) = s(D)$. By Theorem 4.9,

$$-\lambda(K) \geq -\lambda(D) = c(D) - s(D) = 2g(K) - 1\,,$$

where the last equality comes from Seifert's algorithm.

Next, it is known by Yokota [Yo] that for K positive, an with F being the Kauffman polynomial,

$$\min\deg_a F(K) = 2g(K)\,. \tag{6.17}$$

Then, with the known bound[2] (see [FT, Fe, Ta, JLS])

$$\lambda(K) \geq -\min\deg_a F(K) + 1\,, \tag{6.18}$$

we have

$$-\lambda(K) \geq 2g(K) - 1 = \min\deg_a F(K) - 1 \geq -\lambda(K)\,.$$

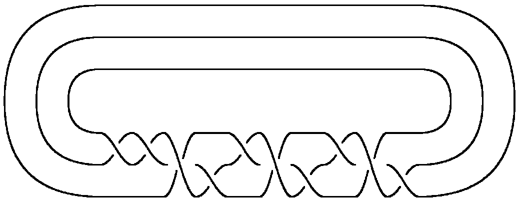
This gives that $\lambda(K) = 1 - 2g(K)$, and finally $\tau(K) = g(K)$ when K is positive (or more generally strongly quasipositive). □

Using joint work of the second author with T. Ito, one can extend this corollary to certain (type II) almost positive knots. (This can be proved by formulas of Rutherford [Rt].) But we also know that strong quasipositivity is not sufficient for (6.16).

Example 6.6 Consider $K = 16_{1379216}$, the closure of the 3-braid

`1 1 [13] 2 1 [13] 2 1 [13] 2,`

[2] We normalize F here so that for the right-hand trefoil, $\min\deg_a F = 2$.



$16_{1379216}$

It has $\min \deg_a F(K) = 7$ (and $g_4(K) = 4$), thus by (6.18) we can conclude that (6.16) fails even for strongly quasipositive K. (This is the only strongly quasipositive example K up to 16 crossings with $\min \deg_a F(K) < 2g_4(K)$, so it underscores the value of the tabulation reported in [St2, Appendix].)

A further series of instances satisfying (6.16), which will play a special role in [St3], and were considered also in [Tr], are slice knots K with

$$\lambda(K) = 1\,. \tag{6.19}$$

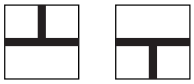
They can be suspected to be quasipositive. But for (6.16) quasipositivity not necessary, as shows the below example.

Example 6.7 The knot $K = 12_{1628}$ has $\lambda = 1$ and $\tau = 0$ (see [LvM]), thus were it to be quasipositive, it would have $\tau = g_4 = 0$, so it would be slice. But this is easily ruled out from the Milnor-Fox condition; the determinant $\det(12_{1628}) = 17$ is not a square.

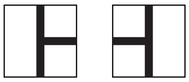
7 Conclusion

The work described here started with the simple question: how does a braided surface of Euler characteristic 0 look like? While there seems little hope to give a classification result, the attempt unfolded a connection into a variety of issues. We encountered many suggestive but difficult to resolve questions, whose examination would require deepening this consideration.

For smaller Euler characteristic, one obtains instead of a grid diagram a "grid-embedded (trivalent) graph". It can be described as a PL spatial embedding of a trivalent graph whose diagram can be built up with the tiles in (2.17), and the two extra tiles



but *not*



Developing a similar theory of grid-embedded graphs will thus also be a long – but nevertheless perhaps very interesting – undertaking.

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