Strong quasipositivity, Thurston-Bennequin invariants, and arc index

Alexander Stoimenow,

(mostly) joint w/
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This talk is dedicated to the memory of my coauthor and friend Hwa Jeong Lee

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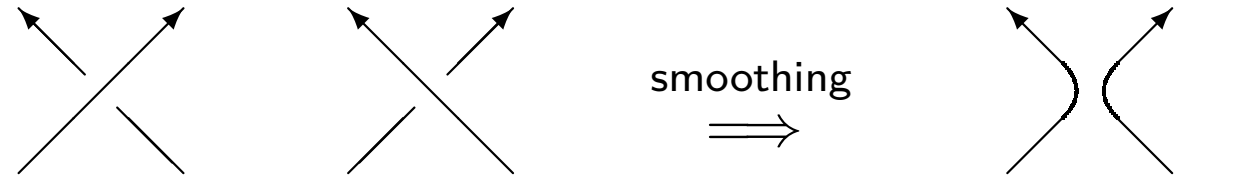
1. Introduction, Definitions etc.

Link/Knot Diagrams

All link diagrams and links are assumed oriented.

Crossings in an oriented diagram D of a knot K are called



positive D_+ negative D_- smoothed out D_0 (1)

$c_\pm(D)$: number of positive, resp. negative crossings of D

$c(D) = c_+(D) + c_-(D)$: number of crossings in D.

$w(D) = c_+(D) - c_-(D)$: writhe of D.

$s(D)$: number of circles after smoothing all crossings of D.
(Such circles are called Seifert circles.)

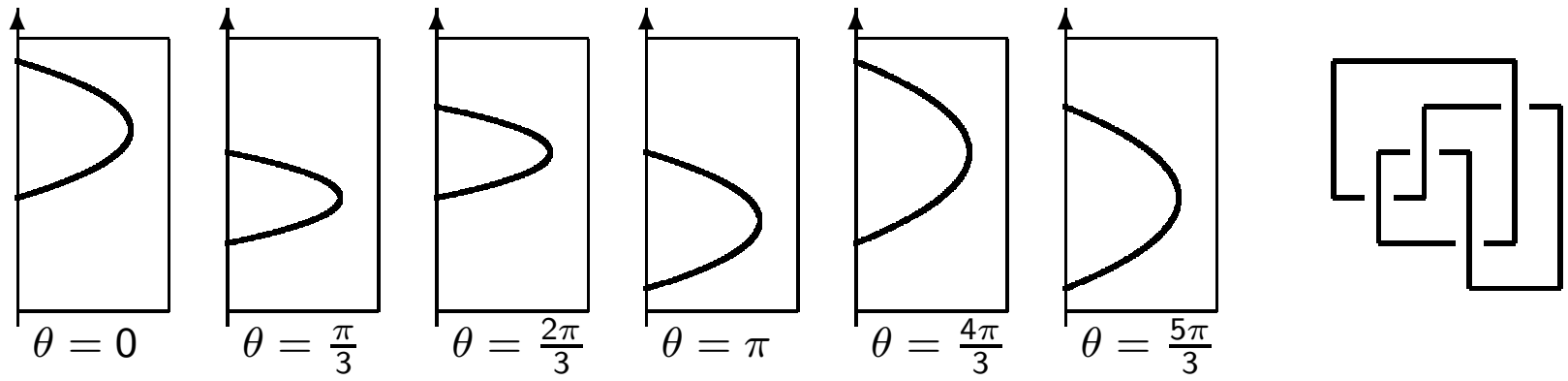
$c(K)$: crossing number of K
$c(K) = \min\{c(D) \mid D \text{ is a diagram of } K\}$

Definition 1

$!D, !K$ mirror image. *If* $K = !K$, *then* K *is* achiral *(or amphicheiral).*

Arc presentation and arc index

An **arc presentation** of a knot or a link L is an ambient isotopic image of L contained in the union of finitely many half planes, called **pages**, with a common boundary line in such a way that each half plane contains a properly embedded single arc.



The minimal number of pages among all arc presentations of a link L is called the **arc index** of L and is denoted by $\boldsymbol{a(L)}$.

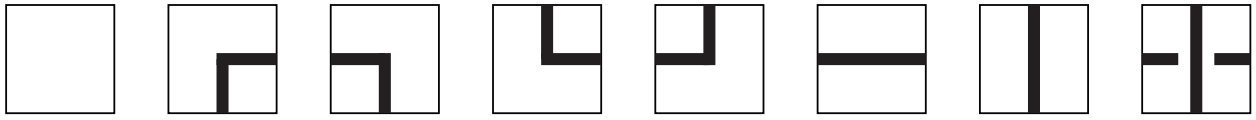
Remark 2

$a(L_1 \# L_2) = a(L_1) + a(L_2) - 2$ *(Cromwell)*

Grid diagram and arc presentation

A **grid diagram** is a knot diagram which is composed of finitely many horizontal edges and the same number of vertical edges such that vertical edges always cross over horizontal edges.

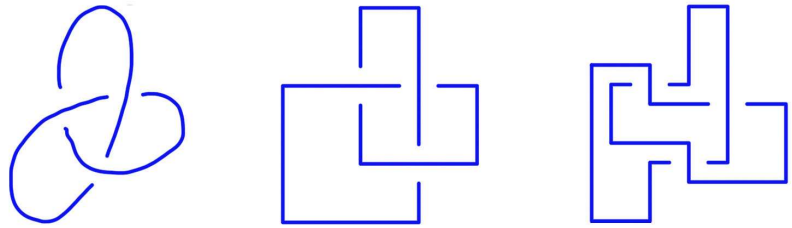
Up to adjustment, grid diagram $\Longleftrightarrow$
mosaic that can be composed in the plane by the tiles



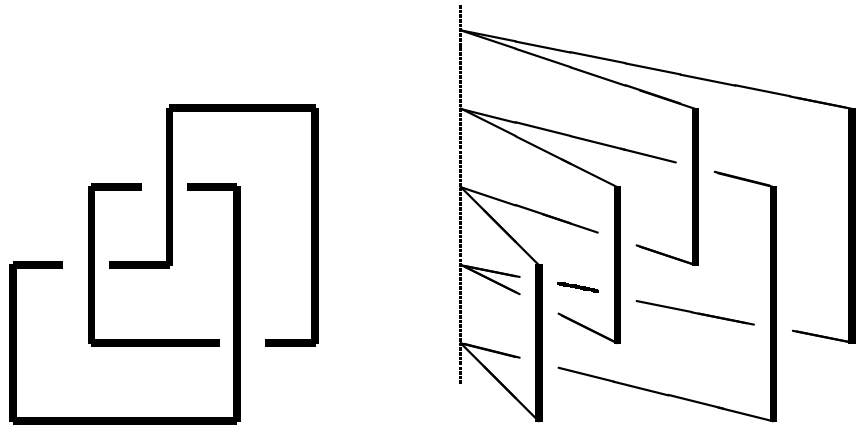
(2)

(à la Kauffman-Lomonaco, 和靜-. . .)

It is not hard to see that every knot admits a grid diagram.

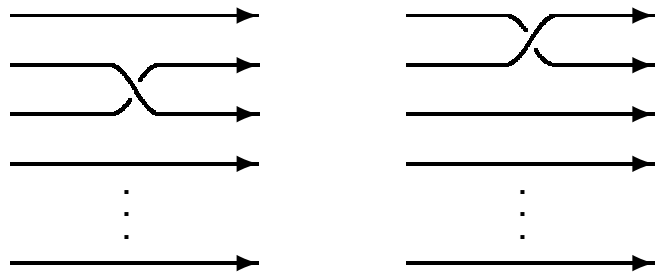


The figure below explains that arc presentations and grid diagrams correspond one-to-one.



Braids and the Artin Generators $\sigma_1, \ldots, \sigma_{n-1}$

For $n \geq 2$, the **braid group** B_n has **Artin generators** $\sigma_1, \ldots, \sigma_{n-1}$,



σ_2 $\qquad\qquad$ σ_1^{-1}

and relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad 1 \leq i \leq n-2,$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i, \qquad |i-j| \geq 2.$$

Closure $\hat{\beta} = L$ for $\beta \in B_n$ can be regarded as a diagram of L, for a *word* of β in σ_i.
Then $w(\hat{\beta}) =$ *exponent sum* of β, and $s(\hat{\beta}) = n$ number of strings.

Quasipositive and Strongly Quasipositive Braids/Links

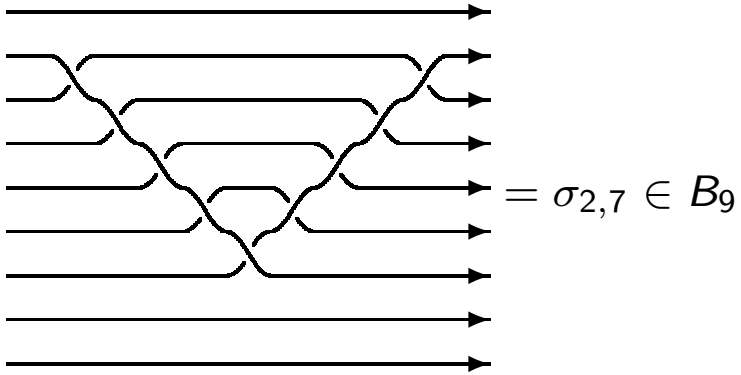
Definition 3

A a braid $\beta \in B_n$ and its closure $\hat{\beta} = L$ are called ***quasipositive*** *(Rudolph, Boileau-Orevkov, etc.) if β is of the form*

$$\beta = \prod_{k=1}^{l} w_k \sigma_{i_k} w_k^{-1}, \tag{3}$$

which is product of conjugates of σ_i's (but not *their inverses).*
A braid β and its closure $\hat{\beta} = L$ are called ***strongly quasipositive*** *if β is of the form as in (3) with*

$$w_k \sigma_{i_k} w_k^{-1} = \sigma_i \ldots \sigma_{j-2} \sigma_{j-1} \sigma_{j-2}^{-1} \ldots \sigma_i^{-1} = \sigma_{i,j} \tag{4}$$



Band Representation of Braids

Notice that $\sigma_{i,i+1} = \sigma_i$. B_n is generated by $\sigma_{i,j}, 1 \leq i < j \leq n$, which are called the **band generators**.
More specifically, a braid representative

$$\beta = \prod_{k=1}^{l} \sigma_{i_k,j_k}^{\pm 1} \tag{5}$$

is called a **band representation** (Birman-Ko-Lee).The closure $L = \hat{\beta}$ of a band representation β naturally spans a Seifert surface.

A word (5) is a **band representation** and its Seifert surface is a **braided surface** (Rudolph '83) of $L = \hat{\beta}$. A minimal genus braided surface is called **Bennequin surface** (Birman-Menasco '91).

Positive Links

Definition 4

A link diagram D is **positive** *if $c_-(D) = 0$ (i.e., $w(D) = c(D)$). A link L is positive if it has a positive diagram.*

It is known that

$$\{\text{quasipositive}\} \supset \{\text{strongly quasipositive}\} \overset{\text{Rudolph, Nakamura}}{\supset} \{\text{positive}\},$$

and the inclusions are proper.

Example 5

Torus links $T_{p,n} = T(p,n) =$ *closure of* $(\sigma_1 \cdots \sigma_{p-1})^n \in B_p$ *(see later §5) are positive.*

We also say D, L are *negative*, if $!D, !L$ are positive.

Bennequin's inequality

Theorem 6 (Bennequin '84)

For $\hat{\beta} = L$, we have $-\chi(L) \geq w(\hat{\beta}) - s(\hat{\beta})$

Corollary 7

Every strongly quasipositive surface is a Bennequin surface.

Definition 8

L is Bennequin-sharp *if*

$$-\chi(L) \;=\; \max_{\hat{\beta}=L} w(\hat{\beta}) - s(\hat{\beta})$$

Corollary 9

L is strongly quasipositive $\Longrightarrow$ L is Bennequin-sharp.

Problem 10 (Bennequin sharpness problem)

L is strongly quasipositive $\Longleftrightarrow$ L is Bennequin-sharp.

Braid Indices

Definition 11

- *Let* $\boldsymbol{b(L)}$ *be the* ***braid index*** *of L, the minimal number of strings of a braid representative of L.*
- *Let* $\boldsymbol{b_b(L)}$ *be the* ***Bennequin braid index*** *of L, the minimal number of strings to span a Bennequin surface of L.*
- *When L is strongly quasipositive, let* $\boldsymbol{b_{sqp}(L)}$ *be the minimal number of strings to span a strongly quasipositive surface of L.*

$$b_{sqp}(L) \geq b_b(L) \geq b(L)$$

Example 12 (Hirasawa-S. '03)

$b_b(K) > b(K)$ *can occur.*

Question 13 (Rudolph)

Is $b_{sqp}(L) = b(L)$ *for every strongly quasipositive link L?*

It is true, e.g., for all prime s.q.p. knots up to 16 crossings.

HOMFLY-PT polynomial

$$P : \{\text{oriented links}\} \to \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$$

Convention (Morton), in (1):

$$v^{-1}P(D_+) - vP(D_-) = zP(D_0)\,, \qquad P(\bigcirc) = 1\,. \tag{6}$$

Powers of v, z are always odd/even for even/odd components of L.

Question 14 (Jones '85)

$P(K) = 1 \implies K = \bigcirc\,?$

Example 15

If a knot K is quasipositive and achiral ($K = !K$) then

$P(K) = 1 \overset{Qu.\ 14}{\Longrightarrow} K = \bigcirc$.

Theorem 16 (Morton-Franks-Williams)

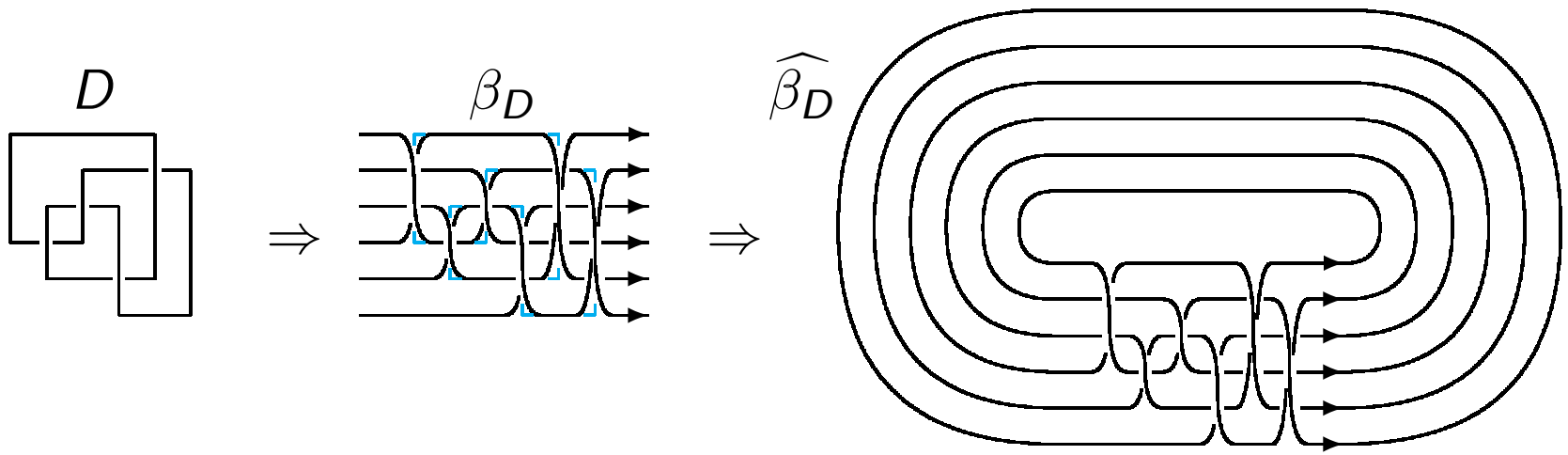
$b(L) \geq 1 + \frac{1}{2}\,\text{span}_{\,v} P(L)\,.$

2. grid diagram $\Longleftrightarrow$ band representation

Definition 17

For a knot K and an integer t, let $\boldsymbol{A(K,t)}$ *be the boundary of the t-framed K-knotted annulus. (Convention: $t =$* linking number*.)*

Nutt's construction: Let D be a grid diagram of a knot K. Replacing each vertical segment with a half twisted band as shown below, we get a braid in band representation, denoted by β_D. Then the closure $\widehat{\beta_D}$ bounds a twisted annulus. Therefore $\widehat{\beta_D} = A(K, \lambda(D))$ for some integer $\lambda(D)$.



Definition 18

*The **Thurston-Bennequin invariant** of a grid diagram D is defined by the formula*

$$TB(D) = w(D) - \#(\text{NW-corners}(D))$$

Lemma 19

For any grid diagram D of K,

$$\lambda(D) = -TB(D)\,. \tag{7}$$

Let $TB(K)$ be the **maximal Thurston-Bennequin invariant** of K and

$$\lambda(K) := \min\{\ t \mid A(K,t) \text{ is strongly quasipositive }\}$$

.

Corollary 20 (Rudolph)

(For $K \neq$ unknot $\bigcirc$) $\lambda(K) = -TB(K)$ *and*
$A(K,t)$ *is strongly quasipositive* $\iff \quad t \geq -TB(K)$.

Remark 21

$\lambda(\bigcirc) = 0$ *but* $-TB(\bigcirc) = 1$. *(But the unknot can be easily dealt with everywhere below on an* ad hoc *basis.)*

Theorem 22 (Dynnikov-Prasolov)

If D is a minimal (i.e., size $a(K)$) grid diagram of $K(\neq \bigcirc)$, then $\lambda(D) = \lambda(K)$.

Corollary 23

$\lambda(K) + \lambda(!K) = a(K)$. *($! =$ mirror image)*

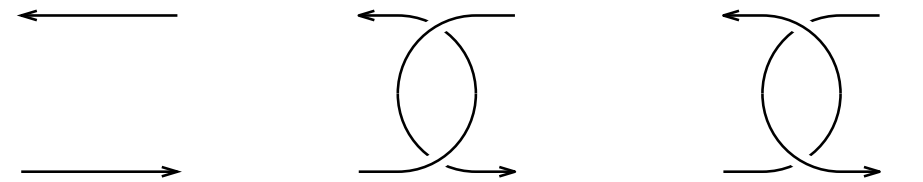
Remark 24

Can reproduce (and improve) Ohyama's ('93) braid index inequality.

3. (Strong) quasipositivity of Whitehead doubles

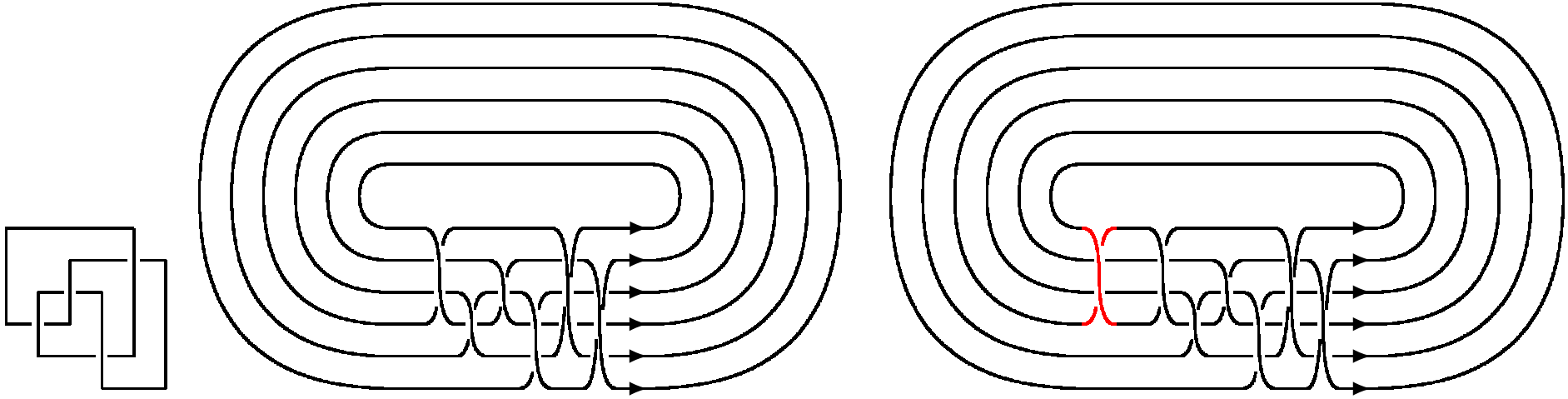
Definition 25

Whitehead double $\boldsymbol{W_{\pm}(K,t)}$ *of a knot* K *with framing* t *and positive/negative clasp.*



$A(K,t)$ $W_-(K,t)$ $W_+(K,t)$

Example 26



4_1 $A(4_1,3)$ $W_+(4_1,3)$

Question 27

When is $W_{\pm}(K,t)$ strongly quasipositive?

We use Corollary 20 and Rudolph's result on Murasugi sum to prove

Theorem 28

$W_{-}(K,t)$ is never strongly quasipositive.
$W_{+}(K,t)$ is strongly quasipositive $\iff$ $t \geq -TB(K)$.

Quasipositivity

We have no complete decision, but general results.

Theorem 29 (S.)

For K alternating, each $W_+(K,t)$ is quasipositive $\iff$ it is strongly quasipositive.

(Uses ℓ-sharpness, below, Casson-Gordon-Gilmer – with thanks to Chuck Livingston – and Hedden's evaluation of the jump number for the τ invariant.)

Example 30 (S.-Orevkov)

$W_+(K,0)$ can be quasipositive but not strongly quasipositive for some (non-alternating) knots K.

We do not know if any $W_-(K,t)$, except

$$K = \bigcirc, \quad t = 0 \tag{8}$$

can be quasipositive.

But:

Theorem 31 (S.)

(a) For every knot K, there is at most one t so that $W_-(K,t)$ is quasipositive.

(b) If $K \neq \bigcirc$ is alternating, negative, or quasipositive and not slice ($\Longleftarrow$ positive), then $\nexists\, t$.

(c) For any composite K with an alternating or positive knot factor, $\nexists\, t$.

(d) If K is achiral and $W_-(K,t)$ is quasipositive, then $t = 0$ and $P(W_-(K,t)) = 1$ (P= HOMFLY-PT polynomial) $\overset{\text{Qu. 14}}{\Longrightarrow}$ (8).

Computation 32

For which "low" ≤ 16 crossing knots K can $W_-(K,t)$ be quasipositive? Down to 18 (prime) knots K and $t = 0$. Five of them are pretzel $K = \mathcal{P}(k,3,-3)$ ($k = 6,\ldots,10$).

Jump numbers

We can obtain the Livingston-Naik ('06) result for jump numbers of slice-torus invariants.

Application to Bennequin-sharpness Problem 10:

Corollary 33

When K is a positive knot, then every $W_{\pm}(K,t)$ is strongly quasipositive $\iff$ it is Bennequin-sharp

Remark 34

This is also true for alternating K, but for (somewhat) different reasons (Proposition 42). These later arguments apply for $A(K,t)$ as well, which does not work here because the "slice-torus" business restricts to knots.

4. HOMFLY-PT polynomial and ℓ-invariant

Assume convention (6).

Definition 35 (width of "the pan"; trim the panhandle)

K knot.

$$\ell(K) := 1 + \frac{1}{2} \operatorname{span}_v P(A(K,t))$$

when t is chosen so that

$$\min \deg_v P(A(K,t)) < 0 < \max \deg_v P(A(K,t))$$

(Keep in mind: powers of v, z are always odd/even for even/odd components.)

Theorem 36

We have a lower bound

$$\ell(K) \leq a(K)\,. \tag{9}$$

Question 37

Is $\ell(K_1 \# K_2) = \ell(K_1) + \ell(K_2) - 2$*? (Compare with Remark 2.)*

Considering the Morton-Beltrami lower bound from the Kauffman polynomial $F(a, z)$:

$$\mathrm{MB}\,(K) := \mathrm{span}_a\, F(K) + 2 \leq a(K)\,, \tag{10}$$

we *conjecture* that ℓ is better.

Conjecture 38

$$\mathrm{MB}\,(K) \leq \ell(K)\,. \tag{11}$$

Conjecture 38 is true for

- prime knots K up to 16 crossings,
- torus knots (Section 5),
- alternating knots (as an equality; in a moment).

Definition 39

K is ℓ-sharp if $\ell(K) = a(K)$.

For prime knots K up to 12 crossings, K is ℓ-sharp except for 10_{132}, four 11 crossing knots 11_{379}, 11_{424}, 11_{455}, 11_{459}, and 21 further examples of 12 crossings.

For an *alternating* knot K, the inequality (10) is exact, so the following "fits" with Conjecture 38.

Proposition 40 (Diao-Morton(-...) '24)

Alternating knots are ℓ-sharp.

This motivates us to study ℓ-sharpness.
Among applications, we have our generalization of Diao-Morton:

Proposition 41

Assume K is a non-trivial ℓ-sharp knot. Then

$$b(A(K,t)) = \begin{cases} \lambda(K) - t & \text{if } t \leq \lambda(K) - a(K) \\ a(K) & \text{if } \lambda(K) - a(K) \leq t \leq \lambda(K) \\ t - \lambda(K) + a(K) & \text{if } t \geq \lambda(K) \end{cases}$$

Proposition 42

If K is ℓ-sharp, then Problem 10 and Question 13 are resolved affirmatively for all $L = A(K,t)$ and $W_{\pm}(K,t)$.

Also $b_b(L) = b(L)$, except possibly *one* t for $L = W_-(K,t)$.
Difficult examples: Computation 32, incl. $W_-(\mathcal{P}(k,3,-3),0)$ ...
But: can be avoided when K is alternating.

TB invariant estimate

For $\lambda(K)$, we have (Fuchs-Tabachnikov, Ferrand, T. Tanaka, ...)

$$\lambda(K) \geq \mathrm{FT}(K) := 1 - \min \deg_a F(K) \quad \binom{\text{also fixes our mirroring}}{\text{convention for } F} \tag{12}$$

Note: compatibly with Corollary 23,

$$\mathrm{FT}(K) + \mathrm{FT}(!K) = \mathrm{MB}(K).$$

Proposition 43 ("Frying eggs in the pan")

We can determine a number $\theta(K)$ from $P(A(K,t))$ so that

$$\theta(K) \leq \lambda(K) \leq \theta(K) + a(K) - \ell(K), \qquad \theta(K) + \theta(!K) = \ell(K)$$

ℓ-invariant version of (12);
in particular $\theta(K) = \lambda(K)$ when K is ℓ-sharp.

The following refines Conjecture 38:

Conjecture 44

$$\mathrm{FT}(K) \leq \theta(K)\,.$$

So that we have (with purple being expected):

$$\begin{array}{ccccc} \lambda(K) & + & \lambda(!K) & = & a(K) \\ \vee & & \vee & & \vee \\ \theta(K) & + & \theta(!K) & = & \ell(K) \\ \vee & & \vee & & \vee \\ \mathrm{FT}(K) & + & \mathrm{FT}(!K) & = & \mathrm{MB}(K) \end{array}$$

Conjecture 44 is true for

- prime knots K up to 16 crossings,
- mirrored (i.e., negative) torus knots,
- positive (incl. torus) knots (as an equality),
- alternating knots (as an equality).

Further methods and computations

We can further use (truncated!) degree-3-satellite HOMFLY and 2-cable Kauffman polynomials to get practically always sharp lower bounds on $a(K)$, $\lambda(K)$, ...

Arc indices and maximal TB invariants for 13 crossing knots are now available on KnotInfo (*new site:* https://knotinfo.org) and for 14 crossings on my website.
http://www.stoimenov.net/stoimeno/homepage/ptab/index.html

(10) seems to perform well when $a(K)$ is large compared to $c(K)$. It is always exact in the case $a(K) = c(K) + 2$ ($\iff$ K is alternating). So far there is no example answering negatively this

Question 45

$$c(K) = a(K) \implies \text{(10) is exact } (\overset{\text{Conj. 38}}{\implies} K \text{ is } \ell\text{-sharp})?$$

5. Torus knots

This section is joint with H. Sati & V.Singh, NYU Abu Dhabi, and A. Mironov, Lebedev Institute and Moscow, ITEP.

Recall: $T_{p,n} = T(p,n) =$ closure of $(\sigma_1 \cdots \sigma_{p-1})^n$ (positive)

Assumption: $(p,n) = 1$ (*knot* case) $n > p (> 1)$.

Theorem 46 (Etnyre-Honda)

$a(T(p,n)) = p + n$

Theirs was the only proof, geometric (and '$\leq$' is "easy").

Theorem 47
$\ell(T(p,n)) = p + n$

So now we have a quantum-algebra proof. Then we get all geometric applications we wanted, for example (but not only):

Corollary 48
Let $K = T_{p,n}$ or $K = !T_{p,n}$. And let $L = A(K,t)$ or $L = W_{\pm}(K,t)$ (for some $t \in \mathbb{Z}$). Then

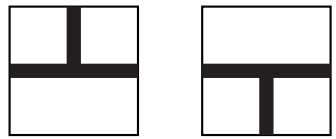
$$L \text{ is quasipositive} \iff L \text{ is strongly quasipositive} \iff$$
$$\iff L \text{ is Bennequin-sharp}$$

Also have some extensions to torus *links* (work in progress)...

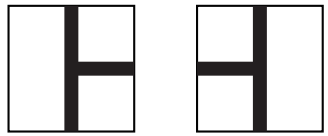
6. Open projects

Problem 49

- *Define ℓ-invariant for links (have some ideas that apply to torus links. . .)*
- *More general ("grid-graph") theory for braided surfaces of lower χ.*
  Grid-graph is PL spatial embedding of a trivalent graph whose diagram can be built up with the tiles in (2)*, and the two extra tiles*



but not



THANK YOU FOR EVERYTHING!